

QUIVERS WITH POTENTIALS ASSOCIATED TO TRIANGULATED SURFACES, PART IV: REMOVING BOUNDARY ASSUMPTIONS

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Dedicated to the memory of Professor Andrei Zelevinsky

ABSTRACT. We prove that the quivers with potentials associated to triangulations of surfaces with marked points, and possibly empty boundary, are non-degenerate, provided the underlying surface with marked points is not a closed sphere with exactly 5 punctures. This is done by explicitly defining the QPs that correspond to tagged triangulations and proving that whenever two tagged triangulations are related by a flip, their associated QPs are related by the corresponding QP-mutation. As a byproduct, for (arbitrarily punctured) surfaces with non-empty boundary we obtain a proof of the non-degeneracy of the associated QPs which is independent from the one given by the author in the first paper of the series.

The main tool used to prove the aforementioned compatibility between flips and QP-mutations is what we have called *Popping Theorem*, which, roughly speaking, says that an apparent lack of symmetry in the potentials arising from ideal triangulations with self-folded triangles can be fixed by a suitable right-equivalence.

CONTENTS

1. Introduction	1
Acknowledgements	3
2. Algebraic and combinatorial background	4
2.1. Automorphisms of complete path algebras	4
2.2. Signatures and weak signatures of tagged triangulations	5
3. Definition of $(Q(\tau), S(\tau, \mathbf{x}))$	9
4. Flip/mutation of folded sides: The problem	12
5. Popped potentials: Definition and flip/mutation dynamics	13
6. Pop is right-equivalence: Statement and proof	16
6.1. Statement	16
6.2. Proof for positive-genus surfaces with empty boundary	17
6.3. Proof for genus-zero surfaces with empty boundary	21
6.4. Proof for surfaces with non-empty boundary	24
7. Leaving the positive stratum	25
8. Flip \leftrightarrow QP-mutation compatibility	28
9. Non-degeneracy of the QPs $(Q(\tau), S(\tau, \mathbf{x}))$	31
10. Irrelevance of signs and scalars attached to the punctures	31
11. Jacobian algebras are finite-dimensional	33
References	34

1. INTRODUCTION

The quivers with potentials associated in [15] to ideal triangulations of arbitrarily punctured surfaces, and independently in [3] in the particular case of unpunctured surfaces, have had appearances both in

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Mathematics (cf. for example [5], [19], [20]) and Physics (cf. for example [1], [2], [8], [23]), with the compatibility between ideal flips and QP-mutations proved in [15] appearing as well.

The definition of the aforementioned QPs is quite simple-minded: every “*sufficiently nice*” ideal triangulation possesses two obvious types of cycles on its signed adjacency quiver, and the associated potential just adds them up. Simplicity has come with a price though, namely, a considerable amount of algebraic-combinatorial computations in order to prove statements that are otherwise quite natural (cf. for example [15]). The price is paid once more in the present note, where to each tagged triangulation τ of a surface with marked points (Σ, \mathbb{M}) we associate a QP $(Q(\tau), S(\tau, \mathbf{x}))$, and prove that whenever two tagged triangulations τ and σ are related by the flip of a tagged arc i , the QPs $\mu_i(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\sigma), S(\sigma, \mathbf{x}))$ are right-equivalent, provided (Σ, \mathbb{M}) is not a closed sphere with exactly 5 punctures. As a direct consequence of this *flip* \leftrightarrow *QP-mutation compatibility*, we establish the non-degeneracy of all the QPs associated to (tagged) triangulations (except in the case just mentioned).

Let us describe the contents of the paper in more detail. In Section 2 we give some background concerning homomorphisms between complete path algebras (Subsection 2.1) and signatures of tagged triangulations (Subsection 2.2). In Section 3 we define a potential $S(\tau, \mathbf{x})$ for each tagged triangulation τ of a punctured surface with respect to a choice $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ of non-zero scalars for the punctures of the surface, and then recall [15, Theorem 30], stated as Theorem 3.6 below, which says that flips between ideal triangulations are always compatible with QP-mutations.

In the somewhat intuitive Section 4, we use an explicit example to illustrate a general problem that arises when trying to prove that flips of folded sides of ideal triangulations are compatible with QP-mutations. In Section 5 we go back to formal considerations. There, for each ideal triangulation τ having a self-folded triangle we define a *popped potential* $W(\tau, \mathbf{x})$ as the result of applying an obvious quiver automorphism, induced by the self-folded triangle, to the potential associated to τ with respect to another choice $\mathbf{y} = (y_p)_{p \in \mathbb{P}} = ((-1)^{\delta_{p,q}} x_p)$ of non-zero scalars, where q is the puncture enclosed in the self-folded triangle (thus in this paper the signs of the scalars attached to the punctures will play some role, at least in the absence of boundary). Based on Theorem 3.6 we then show Lemma 5.5, which says that as long as none of the two arcs contained in a fixed self-folded triangle is ever flipped, the popped potentials $W(\tau, \mathbf{x})$ associated to the ideal triangulations containing the fixed self-folded triangle have the same flip/mutation dynamics possessed by the potentials $S(\tau, \mathbf{x})$. That is, if two ideal triangulations share a self-folded ideal triangle and are related by a flip, then their popped QPs are related by the corresponding QP-mutation.

Section 6 is the technical core of the present work. Its Subsection 6.1 is devoted to state what is instrumentally the main result of the paper, Theorem 6.1, which we call the *Popping Theorem* and says that if (Σ, \mathbb{M}) is not a closed sphere with exactly 5 punctures, then for any ideal triangulation σ with a self-folded triangle, the pop in σ of such self-folded triangle induces right-equivalence, that is, the QP $(Q(\sigma), S(\sigma, \mathbf{x}))$ and the QP $(Q(\sigma), W(\sigma, \mathbf{x}))$ with popped potential are right-equivalent (see (6.5) below). Our first step towards the proof of Theorem 6.1 makes use of Theorem 3.6 and Lemma 5.5 in order to reduce the alluded proof to showing the mere existence of a single ideal triangulation τ with a self-folded triangle whose pop in τ induces right-equivalence. The existence of such a τ is proved in Subsection 6.2 for positive-genus surfaces with empty boundary, and in Subsection 6.3 for genus-zero surfaces with empty boundary—that is, spheres.

Having proved the Popping Theorem for surfaces with empty boundary, in Subsection 6.4 we use restriction of QPs and gluing of disks along boundary components to deduce that it holds as well in the presence of boundary (cf. Proposition 6.13). The key properties of restriction used are the facts that it preserves right-equivalences, and takes potentials of the form $S(\tau, \mathbf{x})$ to potentials of the same form and popped potentials to popped potentials.

Despite its somewhat technical proof, the Popping Theorem easily yields Theorem 7.1, the second main result of the paper, stated and proved in Section 7, which says that flips of folded sides of self-folded triangles are compatible with QP-mutations. Theorem 7.1 is used in Section 8 to deduce our third main result, Theorem 8.1, which states that arbitrary flips of tagged arcs are always compatible with QP-mutation, that is, that any two tagged triangulations related by a flip give rise to quivers with potentials related by the corresponding QP-mutation. Theorem 8.1 has the non-degeneracy of the QPs $(Q(\tau), S(\tau, \mathbf{x}))$ as an immediate consequence. This is the fourth main result of the present note and is stated in Section 9 as Corollary 9.1.

In Section 10 we show that for surfaces with non-empty boundary, the scalars x_p attached to the punctures $p \in \mathbb{P}$, and more importantly, the signs in the potentials $S(\tau, \mathbf{x})$ arising from the weak signatures ϵ_τ , are irrelevant; that is, for any two choices $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ and $\mathbf{y} = (y_p)_{p \in \mathbb{P}}$ of non-zero scalars, the QPs $(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\tau), S(\tau, \mathbf{y}))$ are right-equivalent. This implies that the QPs defined in [9] for surfaces with non-empty boundary are right-equivalent to the ones defined here.

Finally, in Section 11 we state recent results, proved independently by Ladkani (cf. [17]) and Trepode–Valdivieso–Díaz (cf. [22]), that answer a question from the first version of this manuscript on the Jacobi-finiteness of the QPs associated to ideal triangulations of surfaces with empty boundary.

We remind the reader that starting in [15] we have decided not to work with the situation where (Σ, \mathbb{M}) is an unpunctured or once-punctured monogon or digon, an unpunctured triangle, or a closed sphere with less than five punctures (though the 4-punctured sphere has proven important both in cluster algebra theory and representation theory, cf. [4]). In the remaining situations, up to now, potentials have been defined for all ideal triangulations of arbitrarily-punctured surfaces, regardless of emptiness or non-emptiness of the boundary (cf. [15]; and in the unpunctured non-empty boundary case, [3]), and for all tagged triangulations of arbitrarily-punctured surfaces with non-empty boundary (cf. [9]), but not for non-ideal tagged triangulations of (necessarily punctured) surfaces with empty boundary. Also, non-degeneracy has been shown for all arbitrarily-punctured surfaces with non-empty boundary and for all empty-boundary positive-genus surfaces with exactly one puncture (cf. [15]), but not for empty-boundary surfaces with more than one puncture. Furthermore, the compatibility between flips and QP-mutations has not been shown in general for arbitrary flips of tagged triangulations, but only for flips between ideal triangulations (with no extra assumption on the boundary, cf. [15]), and for flips occurring inside certain subsets Ω'_ϵ of Fomin-Shapiro-Thurston’s *closed strata* Ω_ϵ (with the assumption of non-empty boundary, cf. [9]).

Hence, the results presented here strongly improve the results obtained so far regarding both the non-degeneracy question, and the compatibility between flips of tagged triangulations and mutations of quivers with potentials. Indeed, here we give the definition of a potential for any tagged triangulation of any surface (definition completely missing in [15], and missing in [9] for surfaces with empty boundary), and prove the desired compatibility between QP-mutations and flips of tagged triangulations for all surfaces but the 5-punctured sphere (a strong improvement of [9, Theorem 4.4 and Corollary 4.9]). As a byproduct, we obtain a direct proof of the non-degeneracy of the QPs $(Q(\tau), S(\tau, \mathbf{x}))$ for all surfaces (Σ, \mathbb{M}) different from the 5-punctured sphere, with no assumptions on the possible emptiness of the boundary of Σ (in contrast to [15], where, as we said in the previous paragraph, non-degeneracy was shown only for surfaces with non-empty boundary and for positive-genus empty-boundary surfaces with exactly one punctures). Furthermore, the proof given here of the non-degeneracy of the QPs arising from ideal triangulations of surfaces with non-empty boundary is more direct than the one given in [15], precisely because we calculate the potentials corresponding to tagged triangulations and do not appeal to a proof via rigidity.

Some words on the background needed to understand the statements in this note: Since detailed background sections on Derksen-Weyman-Zelevinsky’s QP-mutation theory and Fomin-Shapiro-Thurston’s development of surface cluster algebras have been included in [15] and [9], we have decided not to include similar sections here. The reader unfamiliar with the combinatorial and algebraic background from tagged triangulations and quivers with potentials not provided here is kindly asked to look at [10], [11], [24], [15, Section 2] or [9, Sections 2 and 3].

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I started considering the problem of compatibility between flips and QP-mutations when I was a PhD student at Northeastern University (Boston, MA, USA). As such, I profitted from numerous discussions with Jerzy Weyman and Andrei Zelevinsky. I am deeply grateful to both of them for their teachings.

Professor Andrei Zelevinsky unfortunately passed away a few of days before the present arXiv submission update was posted. No words can express my gratitude towards him for all the teachings, guidance and encouragement I constantly received from him during my Ph.D. studies and afterwards. My admiration for his way of doing Mathematics will always be of the deepest kind.

2. ALGEBRAIC AND COMBINATORIAL BACKGROUND

2.1. Automorphisms of complete path algebras. Here we briefly recall some basic facts and definitions concerning homomorphisms between complete path algebras.

The following proposition tells us that in order to have an R -algebra homomorphism between complete path algebras, it is sufficient to send each arrow a to a (possibly infinite) linear combination of paths with the same starting and ending points as a . It also gives us a criterion to decide whether a given R -algebra homomorphism is an isomorphism using basic linear algebra.

Proposition 2.1 ([10, Proposition 2.4]). *Let Q and Q' be quivers on the same vertex set, and let their respective arrow spans be A and A' . Every pair $(\varphi^{(1)}, \varphi^{(2)})$ of R -bimodule homomorphisms $\varphi^{(1)} : A \rightarrow A'$, $\varphi^{(2)} : A \rightarrow \mathfrak{m}(Q')^2$, extends uniquely to a continuous R -algebra homomorphism $\varphi : R\langle\langle Q \rangle\rangle \rightarrow R\langle\langle Q' \rangle\rangle$ such that $\varphi|_A = (\varphi^{(1)}, \varphi^{(2)})$. Furthermore, φ is R -algebra isomorphism if and only if $\varphi^{(1)}$ is an R -bimodule isomorphism.*

Definition 2.2 ([10, Definition 2.5]). An R -algebra automorphism $\varphi : R\langle\langle Q \rangle\rangle \rightarrow R\langle\langle Q \rangle\rangle$ is said to be

- *unitriangular* if $\varphi^{(1)}$ is the identity of A ;
- *of depth $\ell < \infty$* if $\varphi^{(2)}(A) \subseteq \mathfrak{m}^{\ell+1}$, but $\varphi^{(2)}(A) \not\subseteq \mathfrak{m}^{\ell+2}$;
- *of infinite depth* if $\varphi^{(2)}(A) = 0$.

The depth of φ will be denoted $\text{depth}(\varphi)$.

Thus for example, the identity is the only unitriangular automorphism of $R\langle\langle Q \rangle\rangle$ that has infinite depth. Also, composition of unitriangular automorphisms is unitriangular.

For the following lemma, we use the convention that $\mathfrak{m}^\infty = 0$.

Lemma 2.3 ([10, Equation (2.4)]). *If φ is a unitriangular automorphism of $R\langle\langle Q \rangle\rangle$, then for every $n \geq 0$ and every $u \in \mathfrak{m}^n$ we have $\varphi(u) - u \in \mathfrak{m}^{n+\text{depth}(\varphi)}$.*

Lemma 2.4. *Let Q be any quiver, and $(\psi_n)_{n>0}$ be a sequence of unitriangular R -algebra automorphisms of $R\langle\langle Q \rangle\rangle$. Suppose that $\lim_{n \rightarrow \infty} \text{depth}(\psi_n) = \infty$. Then the limit*

$$\psi = \lim_{n \rightarrow \infty} \psi_n \psi_{n-1} \dots \psi_2 \psi_1$$

is a well-defined unitriangular R -algebra automorphism of $R\langle\langle Q \rangle\rangle$. If, moreover, S and $(S_n)_{n>0}$ are respectively a potential and a sequence of potentials on Q , satisfying $\lim_{n \rightarrow \infty} S_n = 0$ and such that ψ_n is a right-equivalence $(Q, S + S_n) \rightarrow (Q, S + S_{n+1})$ for all $n > 0$, then $\psi = \lim_{n \rightarrow \infty} \psi_n \psi_{n-1} \dots \psi_2 \psi_1$ is a right-equivalence $(Q, S + S_1) \rightarrow (Q, S)$.

Proof. We can suppose, without loss of generality, that $\text{depth}(\psi_n) < \infty$ for all $n > 0$ (this is because the only unitriangular automorphism of $R\langle\langle Q \rangle\rangle$ that has infinite depth is the identity). Since $\lim_{n \rightarrow \infty} \text{depth}(\psi_n) = \infty$, there exists $m_2 > 0$ such that $\text{depth}(\psi_k) > \text{depth}(\psi_1)$ for all $k \geq m_2$. And once we have a positive integer m_n , we can find $m_{n+1} > m_n$ such that $\text{depth}(\psi_k) > \text{depth}(\psi_{m_n})$ for all $k \geq m_n$. Set $\varphi_n = \psi_{m_n} \psi_{m_n-1} \dots \psi_{m_{n-1}+1}$, with the convention that $m_1 = 1$ and $m_0 = 0$. Then $(\text{depth}(\varphi_n))_{n>0}$ is an increasing sequence of positive integers.

Note that the limit $\lim_{n \rightarrow \infty} \varphi_n \varphi_{n-1} \dots \varphi_2 \varphi_1$ is well-defined if and only if the limit $\lim_{n \rightarrow \infty} \psi_n \psi_{n-1} \dots \psi_2 \psi_1$ is well-defined, and if both of these limits are well-defined, then they are equal. In order to show that $\varphi = \lim_{n \rightarrow \infty} \varphi_n \varphi_{n-1} \dots \varphi_2 \varphi_1$ is well-defined it suffices to show that for every $u \in \mathfrak{m}$ and every $d > 0$ the sequence $(u_n^{(d)})_{n>0}$ formed by the degree- d components of the elements $u_n = \varphi_n \dots \varphi_1(u)$ eventually stabilizes as $n \rightarrow \infty$. Notice that $\text{depth}(\varphi_n) \geq n - 1$ for all $n > 0$. From this and Lemma 2.3 we deduce that, for a given $u \in \mathfrak{m}$, there exists a sequence $(v_n)_{n>0}$ such that $v_n \in \mathfrak{m}^n$ and $u_n = u + \sum_{j=1}^n v_j$ for all $n > 0$. From this we see that $u_n^{(d)} = u_{n+1}^{(d)}$ for $n > d$, so the sequence $(u_n^{(d)})_{n>0}$ stabilizes.

For the second statement of the lemma, let V be the topological closure of the K -vector subspace of $R\langle\langle Q \rangle\rangle$ generated by all elements of the form $\alpha_1 \alpha_2 \dots \alpha_d - \alpha_2 \dots \alpha_d \alpha_1$ with $\alpha_1 \dots \alpha_d$ a cycle on Q . Then V is a K -vector subspace of $R\langle\langle Q \rangle\rangle$ as well and two potentials are cyclically-equivalent if and only if their difference belongs to V .

Compositions of right-equivalences is again a right-equivalence, hence φ_n is a right-equivalence $(Q, S + S_{m_n}) \rightarrow (Q, S + S_{m_{n+1}})$ for all $n > 0$. We deduce that

$$\varphi_n \varphi_{n-1} \dots \varphi_2 \varphi_1 (S + S_1) - (S + S_{m_{n+1}}) \in V.$$

The fact that both sequences $(\varphi_n \varphi_{n-1} \dots \varphi_2 \varphi_1(S + S_1))_{n>0}$ and $(-(S + S_{m_{n+1}}))_{n>0}$ are convergent in $R\langle\langle Q \rangle\rangle$ implies that

$$(\varphi_n \varphi_{n-1} \dots \varphi_2 \varphi_1(S + S_1) - (S + S_{m_{n+1}}))_{n>0}$$

converges as well. Since V is closed, we have

$$\varphi(W + S_1) - W = \lim_{n \rightarrow \infty} (\varphi_n \varphi_{n-1} \dots \varphi_2 \varphi_1(S + S_1) - (S + S_{m_{n+1}})) \in V.$$

This finishes the proof of Lemma 2.4. \square

2.2. Signatures and weak signatures of tagged triangulations. In this subsection we review some elementary facts concerning tagged triangulations, their signatures, and the behavior of the latter ones under flips.

Definition 2.5. A *bordered surface with marked points*, or simply a *surface*, is a pair (Σ, \mathbb{M}) , where Σ is a compact connected oriented Riemann surface with (possibly empty) boundary, and \mathbb{M} is a finite set of points on Σ , called *marked points*, such that \mathbb{M} is non-empty and has at least one point from each connected component of the boundary of Σ . The marked points that lie in the interior of Σ are called *punctures*, and the set of punctures of (Σ, \mathbb{M}) is denoted \mathbb{P} . Throughout the paper we will always assume that (Σ, \mathbb{M}) is none of the following:

- a sphere with less than four punctures;
- an unpunctured monogon, digon or triangle;
- a once-punctured monogon or digon.

Here, by a monogon (resp. digon, triangle) we mean a disk with exactly one (resp. two, three) marked point(s) on the boundary.

For the definitions of the following concepts we kindly ask the reader to look at the corresponding reference:

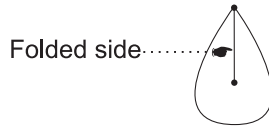
- ordinary arc [11, Definition 2.2];
- compatibility of pairs of ordinary arcs [11, Definition 2.4];
- ideal triangulation [11, Definition 2.6];
- tagged arc [11, Definition 7.1 and Remark 7.3];
- compatibility of pairs of tagged arcs [11, Definition 7.4 and Remark 7.5];
- tagged triangulation [11, Page 111].

The set of ordinary arcs, taken up to isotopy relative to \mathbb{M} , is denoted by $\mathbf{A}^\circ(\Sigma, \mathbb{M})$, while the set of tagged arcs is denoted by $\mathbf{A}^\bowtie(\Sigma, \mathbb{M})$.

Definition 2.6. Let τ be an ideal triangulation of a surface (Σ, \mathbb{M}) .

- (1) For each connected component of the complement in Σ of the union of the arcs in τ , its topological closure \triangle will be called an *ideal triangle* of τ .
- (2) An ideal triangle \triangle is called *interior* if its intersection with the boundary of Σ consists only of (possibly none) marked points. Otherwise it will be called *non-interior*.
- (3) An interior ideal triangle \triangle is *self-folded* if it contains exactly two arcs of τ (see Figure 1).

FIGURE 1. Self-folded triangle



All tagged triangulations of (Σ, \mathbb{M}) have the same cardinality and every tagged arc i in a tagged triangulation τ can be replaced by a uniquely defined, different tagged arc that together with the remaining tagged arcs from τ forms a tagged triangulation σ . This combinatorial replacement will be called *flip*, written $\sigma = f_i(\tau)$. Furthermore, a sequence $(\tau_0, \dots, \tau_\ell)$ of tagged triangulations will be called a *flip-sequence* if τ_{k-1} and τ_k are related by a flip for $k = 1, \dots, \ell$. A flip-sequence will be called *ideal flip-sequence* if it involves only ideal triangulations.

Proposition 2.7. *Let (Σ, \mathbb{M}) be a surface.*

- *Any two ideal triangulations τ and σ of (Σ, \mathbb{M}) are members of an ideal flip-sequence. A flip sequence $(\tau, \tau_1, \dots, \tau_{\ell-1}, \sigma)$ can always be chosen in such a way that $\tau \cap \sigma \subseteq \tau_k$ for all $k = 1, \dots, \ell - 1$.*
- *There is at least one ideal triangulation of (Σ, \mathbb{M}) that does not have self-folded triangles.*
- *Any two ideal triangulations without self-folded triangles are members of an ideal flip-sequence that involves only ideal triangulations without self-folded triangles.*
- *If (Σ, \mathbb{M}) is not a once-punctured surface with empty boundary, then any two tagged triangulations are members of a flip-sequence.*

The first assertion of Proposition 2.7 is well known and has many different proofs, we refer the reader to [18, Pages 36-41] for an elementary one. The second and fourth assertions of the proposition are proved in [11]. A proof of the third assertion can be found in [16, Proof of Corollary 6.7].

Let us recall how to represent ideal triangulations with tagged ones and viceversa.

Definition 2.8 ([11, Definitions 7.2 and 9.2]). Let $\epsilon : \mathbb{P} \rightarrow \{-1, 1\}$ be any function. We define a function $\mathbf{t}_\epsilon : \mathbf{A}^\circ(\Sigma, \mathbb{M}) \rightarrow \mathbf{A}^\times(\Sigma, \mathbb{M})$ that represents ordinary arcs by tagged ones as follows.

- (1) If i is an ordinary arc that is not a loop enclosing a once-punctured monogon, set i to be the underlying ordinary arc of the tagged arc $\mathbf{t}_\epsilon(i)$. An end of $\mathbf{t}_\epsilon(i)$ will be tagged notched if and only if the corresponding marked point is an element of \mathbb{P} where ϵ takes the value -1 .
- (2) If i is a loop, based at a marked point q , that encloses a once-punctured monogon, being p the puncture inside this monogon, then the underlying ordinary arc of $\mathbf{t}_\epsilon(i)$ is the arc that connects q with p inside the monogon. The end at q will be tagged notched if and only if $q \in \mathbb{P}$ and $\epsilon(q) = -1$, and the end at p will be tagged notched if and only if $\epsilon(p) = 1$.

Example 2.9. In Figure 2 we can see all possibilities for $\mathbf{t}_\epsilon(i)$ (resp. $\mathbf{t}_\epsilon(j)$) given an ordinary arc i that

FIGURE 2

i	$\epsilon(p), \epsilon(q)$	$\mathbf{t}_\epsilon(i)$	j	$\epsilon(p), \epsilon(q)$	$\mathbf{t}_\epsilon(j)$
	$\epsilon(p)=1$ $\epsilon(q)=1$			$\epsilon(p)=1$ $\epsilon(q)=1$	
	$\epsilon(p)=1$ $\epsilon(q)=-1$			$\epsilon(p)=1$ $\epsilon(q)=-1$	
	$\epsilon(p)=-1$ $\epsilon(q)=1$			$\epsilon(p)=-1$ $\epsilon(q)=1$	
	$\epsilon(p)=-1$ $\epsilon(q)=-1$			$\epsilon(p)=-1$ $\epsilon(q)=-1$	

does not cut out a once-punctured monogon (resp. a loop j that cuts out a once-punctured monogon), and the values of ϵ at the endpoints of i (resp. at the base-point of j and the puncture enclosed by j). \blacktriangle

To pass from tagged triangulations to ideal ones we need the notion of signature.

Definition 2.10. Let τ be a tagged triangulation of (Σ, \mathbb{M}) .

- (1) Following [11, Definition 9.1], we define the *signature* of τ to be the function $\delta_\tau : \mathbb{P} \rightarrow \{-1, 0, 1\}$ given by

$$(2.1) \quad \delta_\tau(p) = \begin{cases} 1 & \text{if all ends of tagged arcs in } \tau \text{ incident to } p \text{ are tagged plain;} \\ -1 & \text{if all ends of tagged arcs in } \tau \text{ incident to } p \text{ are tagged notched;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\delta_\tau(p) = 0$, then there are precisely two arcs in τ incident to p , the untagged versions of these arcs coincide and they carry the same tag at the end different from p .

- (2) The *weak signature* of τ is the function $\epsilon_\tau : \mathbb{P} \rightarrow \{-1, 1\}$ defined by

$$(2.2) \quad \epsilon_\tau(p) = \begin{cases} 1 & \text{if } \delta_\tau(p) \in \{0, 1\}; \\ -1 & \text{otherwise.} \end{cases}$$

Definition 2.11 ([11, Definition 9.2]). Let τ be a tagged triangulation of (Σ, \mathbb{M}) . We replace each tagged arc in τ with an ordinary arc by means of the following rules:

- (1) delete all tags at the punctures p with non-zero signature
- (2) for each puncture p with $\delta_\tau(p) = 0$, replace the tagged arc $i \in \tau$ which is notched at p by a loop enclosing p and i .

The resulting collection of ordinary arcs will be denoted by τ° .

The next proposition follows from the results in [11, Subsection 9.1].

Proposition 2.12. *Let (Σ, \mathbb{M}) be a surface.*

- (1) *For every function $\epsilon : \mathbb{P} \rightarrow \{-1, 1\}$, the function $\mathbf{t}_\epsilon : \mathbf{A}^\circ(\Sigma, \mathbb{M}) \rightarrow \mathbf{A}^\bowtie(\Sigma, \mathbb{M})$ is injective and preserves compatibility. Thus, if i_1 and i_2 are compatible ordinary arcs, then $\mathbf{t}_\epsilon(i_1)$ and $\mathbf{t}_\epsilon(i_2)$ are compatible tagged arcs. Consequently, if T is an ideal triangulation of (Σ, \mathbb{M}) , then $\mathbf{t}_\epsilon(T) = \{\mathbf{t}_\epsilon(i) \mid i \in T\}$ is a tagged triangulation of (Σ, \mathbb{M}) . Moreover, if T_1 and T_2 are ideal triangulations such that $T_2 = f_i(T_1)$ for an arc $i \in T_1$, then $\mathbf{t}_\epsilon(T_2) = f_{\mathbf{t}_\epsilon(i)}(\mathbf{t}_\epsilon(T_1))$.*
- (2) *If τ is a tagged triangulation of (Σ, \mathbb{M}) , then τ° is an ideal triangulation of (Σ, \mathbb{M}) and $i \mapsto i^\circ$ is a bijection between τ and τ° .*
- (3) *For every ideal triangulation T , we have $\mathbf{t}_1(T)^\circ = T$, where $\mathbf{1} : \mathbb{P} \rightarrow \{-1, 1\}$ is the constant function taking the value 1.*
- (4) *For every tagged triangulation τ and every tagged arc $i \in \tau$ we have $\mathbf{t}_{\epsilon_\tau}(i^\circ) = i$, where $i^\circ \in \tau^\circ$ is the ordinary arc that replaces i in Definition 2.11. Consequently, $\mathbf{t}_{\epsilon_\tau}(\tau^\circ) = \tau$.*
- (5) *Let τ and σ be tagged triangulations such that $\epsilon_\tau = \epsilon_\sigma$. If $\sigma = f_i(\tau)$ for a tagged arc $i \in \tau$, then $\sigma^\circ = f_{i^\circ}(\tau^\circ)$, where $i^\circ \in \tau^\circ$ is the ordinary arc that replaces i in Definition 2.11. Moreover, the diagram of functions*

$$(2.3) \quad \begin{array}{ccccc} \tau & \xrightarrow{?^\circ} & \tau^\circ & \xrightarrow{\mathbf{t}_{\epsilon_\tau}} & \mathbf{t}_{\epsilon_\tau}(\tau^\circ) \\ \downarrow & & \downarrow & & \downarrow \\ \sigma & \xrightarrow{?^\circ} & \sigma^\circ & \xrightarrow{\mathbf{t}_{\epsilon_\sigma}} & \mathbf{t}_{\epsilon_\sigma}(\sigma^\circ) \end{array}$$

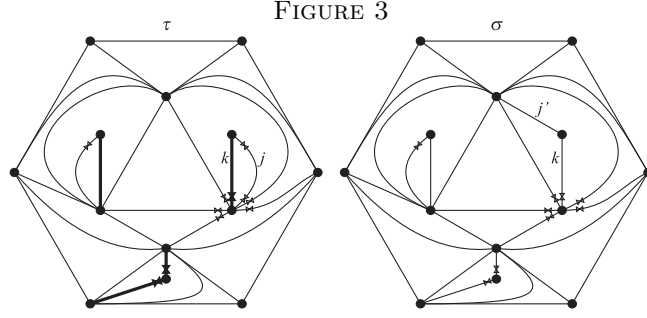
commutes, where the three vertical arrows are canonically induced by the operation of flip.

- (6) *Let τ and σ be tagged triangulations such that $\sigma = f_i(\tau)$ for some tagged arc $i \in \tau$. Then ϵ_τ and ϵ_σ either are equal or differ at exactly one puncture q . In the latter case, if $\epsilon_\tau(q) = 1 = -\epsilon_\sigma(q)$, then i° is a folded side of τ° incident to the puncture q , and $\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma^\circ) = f_{i^\circ}(\mathbf{t}_1(\tau^\circ))$, where $\epsilon_\tau \epsilon_\sigma : \mathbb{P} \rightarrow \{-1, 1\}$ is the function defined by $p \mapsto \epsilon_\tau(p) \epsilon_\sigma(p)$. Moreover, the diagram of functions*

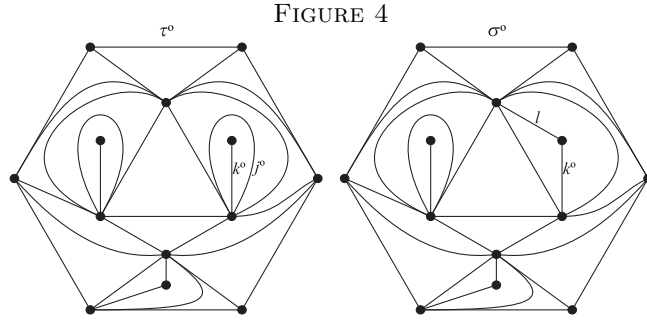
$$(2.4) \quad \begin{array}{ccccc} \tau & \xrightarrow{?^\circ} & \tau^\circ & \xrightarrow{\mathbf{t}_1} & \mathbf{t}_1(\tau^\circ) \\ \downarrow & & \downarrow & & \downarrow \\ \sigma = f_k(\tau) & \xrightarrow{?^\circ} & \sigma^\circ & \xrightarrow{\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}} & \mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma^\circ) = f_{k^\circ}(\mathbf{t}_1(\tau^\circ)) \end{array}$$

commutes, where the two vertical arrows are canonically induced by the operation of flip.

Example 2.13. Consider the tagged triangulation τ depicted on the left side of Figure 3. Given any tagged arc $i \in \tau$, the tagged triangulation $f_i(\tau)$ will satisfy $\epsilon_\tau = \epsilon_{f_i(\tau)}$ if and only if i is not one of the tagged arcs in τ that have been drawn bolder than the rest.

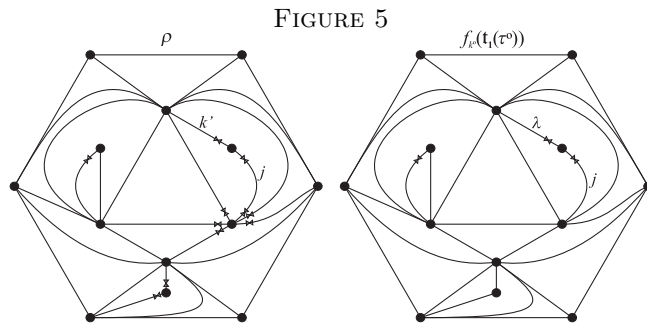


If we flip the tagged arc $j \in \tau$ the resulting tagged triangulation $\sigma = f_j(\tau)$, depicted on the right side of Figure 3, certainly satisfies $\epsilon_\tau = \epsilon_\sigma$. In Figure 4 we have drawn the ideal triangulations τ° and σ° , and the reader can easily verify that, as stated in the fifth assertion of Proposition 2.12, we have $\sigma^\circ = f_{j^\circ}(\tau^\circ)$. The fact that the tagged arc j' in Figure 3 equals the ordinary arc l in Figure 4 is pure coincidence: τ



and σ could as well have been such that the end of j' not incident to k were a notch. What is important is that the functions $?\circ : \tau \rightarrow \tau^\circ$ and $?\circ : \sigma \rightarrow \sigma^\circ$ respectively send j to j° and j' to l , which amounts to the commutativity of the left square in the diagram (2.3) (for tagged triangulations with the same weak signature). Let us stress also the fact that $t_{\epsilon_\tau}(k^\circ) = k = t_{\epsilon_\sigma}(k^\circ)$, $t_{\epsilon_\tau}(j^\circ) = j$, $t_{\epsilon_\sigma}(l) = j'$, which amounts to the commutativity of the right square of the diagram (2.3) (provided $\epsilon_\tau = \epsilon_\sigma$, of course).

If we on the other hand flip the tagged arc $k \in \tau$, we obtain the tagged triangulation $\rho = f_k(\tau)$ depicted on the left side of Figure 5. On the right side of this Figure we can see the tagged triangulation $f_{k^\circ}(t_1(\tau^\circ))$,



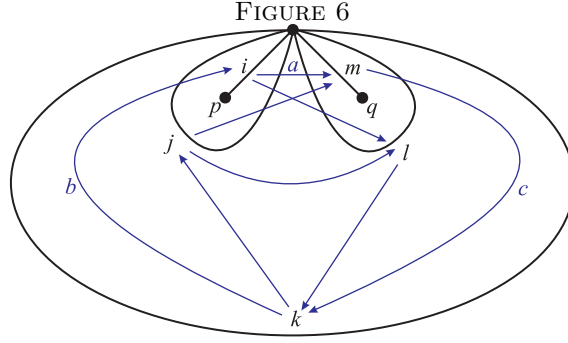
and the equality $t_{\epsilon_\tau \epsilon_\rho}(\rho^\circ) = f_{k^\circ}(t_1(\tau^\circ))$, stated in the fifth assertion of Proposition 2.12, is easily verified. Here again, the fact that the tagged arc $k' \in \rho$ equals the tagged arc $\lambda \in t_{\epsilon_\tau \epsilon_\rho}(\rho^\circ) = f_{k^\circ}(t_1(\tau^\circ))$ is mere coincidence. What matters is that the diagram (2.4) commutes. \blacktriangle

3. DEFINITION OF $(Q(\tau), S(\tau, \mathbf{x}))$

Let (Σ, \mathbb{M}) be a surface, with puncture set \mathbb{P} . No assumptions on the boundary of Σ are made throughout this section. In what follows, $\widehat{Q}(\tau)$ will denote the *unreduced signed-adjacency quiver* of τ (cf. [15, Definition 8]) and $Q(\tau)$ will denote the *signed-adjacency quiver* of τ (which can be obtained from $\widehat{Q}(\tau)$ by deleting all 2-cycles).

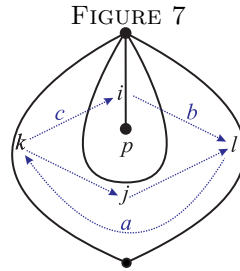
Definition 3.1. Let τ be an ideal triangulation of (Σ, \mathbb{M}) , and $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ be a choice of non-zero scalars (one scalar $x_p \in K$ per puncture p).

- Each interior non-self-folded ideal triangle \triangle of τ gives rise to an oriented triangle $\alpha\beta\gamma$ of $\widehat{Q}(\tau)$, let $\widehat{S}^\triangle(\tau) = \alpha\beta\gamma$ be such oriented triangle up to cyclical equivalence.
- If the interior non-self-folded ideal triangle \triangle with sides j, k and l , is adjacent to two self-folded triangles like in the configuration of Figure 6, define $\widehat{U}^\triangle(\tau, \mathbf{x}) = x_p^{-1}x_q^{-1}abc$ (up to cyclical equivalence), where p and q are the punctures enclosed in the self-folded triangles adjacent to \triangle . Otherwise, if it is adjacent to less than two self-folded triangles, define $\widehat{U}^\triangle(\tau, \mathbf{x}) = 0$.



lence), where p and q are the punctures enclosed in the self-folded triangles adjacent to \triangle . Otherwise, if it is adjacent to less than two self-folded triangles, define $\widehat{U}^\triangle(\tau, \mathbf{x}) = 0$.

- If a puncture p is adjacent to exactly one arc i of τ , then i is the folded side of a self-folded triangle of τ and around i we have the configuration shown in Figure 7. In case both k and l are indeed arcs



of τ (and not part of the boundary of Σ), we define $\widehat{S}^p(\tau, \mathbf{x}) = -x_p^{-1}abc$ (up to cyclical equivalence). Otherwise, if either k or l is a boundary segment, we define $\widehat{S}^p(\tau, \mathbf{x}) = 0$.

- If a puncture p is adjacent to more than one arc, delete all the loops incident to p that enclose self-folded triangles. The arrows between the remaining arcs adjacent to p form a unique cycle $a_1^p \dots a_{d_p}^p$, without repeated arrows, that exhausts all such remaining arcs and gives a complete round around p in the counter-clockwise orientation defined by the orientation of Σ . We define $\widehat{S}^p(\tau, \mathbf{x}) = x_p a_1^p \dots a_{d_p}^p$ (up to cyclical equivalence).

Definition 3.2. Let τ be a tagged triangulation of (Σ, \mathbb{M}) , and $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ be a choice of non-zero scalars (one scalar $x_p \in K$ per puncture p).

- (1) The *unreduced potential* $\widehat{S}(\tau, \mathbf{x}) \in R\langle\langle \widehat{Q}(\tau) \rangle\rangle$ associated to τ with respect to the choice $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ is:

$$(3.1) \quad \widehat{S}(\tau, \mathbf{x}) = \mathbf{t}_{\epsilon_\tau} \left(\sum_{\Delta} \left(\widehat{S}^\Delta(\tau^\circ) + \widehat{U}^\Delta(\tau^\circ, \mathbf{x}) \right) + \sum_{p \in \mathbb{P}} \left(\epsilon_\tau(p) \widehat{S}^p(\tau^\circ, \mathbf{x}) \right) \right)$$

where the first sum runs over all interior non-self-folded triangles of τ° , and $\epsilon_\tau : \mathbb{P} \rightarrow \{-1, 1\}$ is the weak signature of τ .

- (2) We define $(Q(\tau), S(\tau, \mathbf{x}))$ to be the reduced part of $(\widehat{Q}(\tau), \widehat{S}(\tau, \mathbf{x}))$.

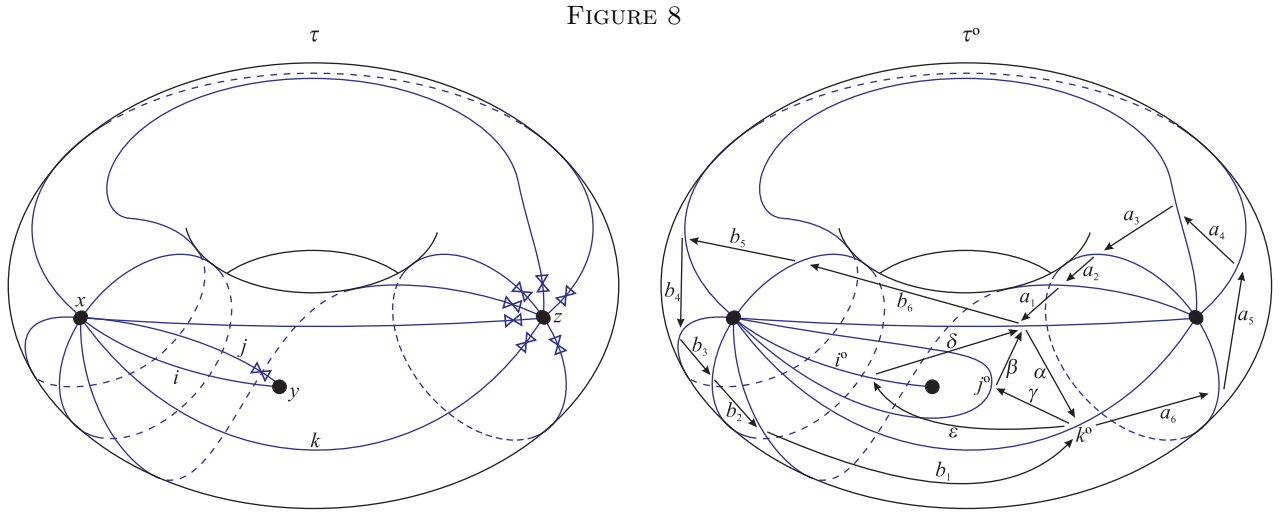
Remark 3.3. (1) If τ is an ideal triangulation, the QP $(Q(\tau), S(\tau, \mathbf{x}))$ defined above coincides with the one defined in [15].

- (2) If τ is a non-ideal tagged triangulation of a punctured surface with non-empty boundary, the QP $(Q(\tau), S(\tau, \mathbf{x}))$ defined above does not coincide with the one defined in [9]. It will be shown later that the two alluded QPs are right-equivalent.

- (3) If τ is an ideal triangulation, the only situation where one needs to apply reduction to $(\widehat{Q}(\tau), \widehat{S}(\tau, \mathbf{x}))$ in order to obtain $S(\tau, \mathbf{x})$ is when there is some puncture incident to exactly two arcs of τ . The reduction is done explicitly in [15, Section 3].

Let us illustrate Definition 3.2 with a couple of examples.

Example 3.4. On the left side of Figure 8 we can see a three-times-punctured torus (with no boundary), with the scalars x_p , $p \in \mathbb{P}$, labeling the three punctures, and a tagged triangulation τ of it. On the right side we can see the ideal triangulation τ° and the quiver $Q(\tau^\circ)$ drawn of the surface.



The function $\mathbf{t}_{\epsilon_\tau} : \mathbf{A}^\circ(\Sigma, \mathbb{M}) \rightarrow \mathbf{A}^{\bowtie}(\Sigma, \mathbb{M})$ restricts to a bijection $\mathbf{t}_{\epsilon_\tau} : \tau^\circ \rightarrow \tau$ acting by

$$\mathbf{t}_{\epsilon_\tau} : i^\circ \mapsto i, \quad j^\circ \mapsto j,$$

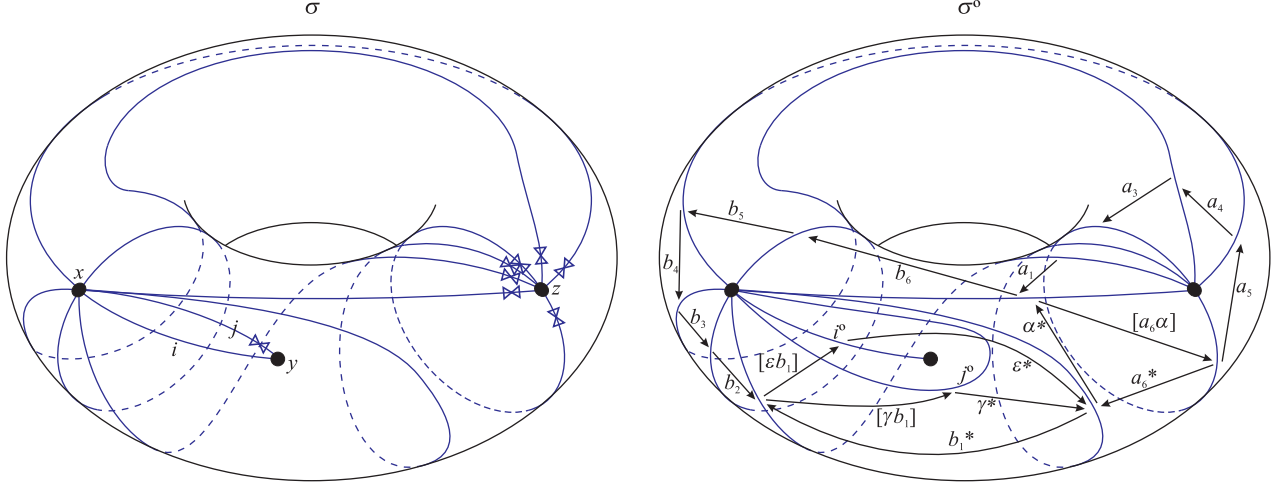
and in the obvious way on the rest of the arcs of τ° . Since the 2-acyclic quiver $Q(\tau)$ clearly coincides with the unreduced quiver $\widehat{Q}(\tau)$, Definition 3.2 tells us that

$$S(\tau, \mathbf{x}) = \alpha\beta\gamma + a_1b_2b_6 + a_2a_6b_1 + a_3b_4a_5 + a_4b_5b_3 + x\delta\epsilon b_1b_2b_3b_4b_5b_6 - y^{-1}\alpha\delta\epsilon - z\alpha a_1a_2a_3a_4a_5a_6.$$

▲

Example 3.5. Let us flip the tagged arc k of the tagged triangulation τ from the previous example. The tagged triangulation $\sigma = f_k(\tau)$, its ideal counterpart σ° , and the quiver $Q(\sigma^\circ)$ are depicted in Figure 9.

FIGURE 9



Just as in the previous example, we have the bijection $t_{\epsilon_\sigma} : \sigma^\circ \rightarrow \sigma$ acting by

$$t_{\epsilon_\sigma} : i^\circ \mapsto i, \quad j^\circ \mapsto j,$$

and in the obvious way on the rest of the arcs of σ° . Since the 2-acyclic quiver $Q(\sigma)$ clearly coincides with $\widehat{Q}(\sigma)$, applying Definition 3.2 we obtain

$$\begin{aligned} S(\sigma, \mathbf{x}) &= \gamma^*[\gamma b_1]b_1^* + a_6^*[a_6\alpha]\alpha^* + a_1b_2b_6 + a_3b_4a_5 + a_4b_5b_3 + x\alpha^*\varepsilon^*[\varepsilon b_1]b_2b_3b_4b_5b_6 - y^{-1}\varepsilon^*[\varepsilon b_1]b_1^* \\ &\quad - z a_1b_1^*a_6^*a_3a_4a_5[a_6\alpha]. \end{aligned}$$

▲

Now we recall one of the main results of [15].

Theorem 3.6 ([15, Theorem 30]). *Let τ and σ be ideal triangulations of (Σ, \mathbb{M}) . If $\sigma = f_k(\tau)$ for some arc $k \in \tau$, then $\mu_k(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\sigma), S(\sigma, \mathbf{x}))$ are right-equivalent QPs.*

Remark 3.7. Note that in the situation of Theorem 3.6, the fact that $\sigma = f_k(\tau)$ is an ideal triangulation implies that the arc k cannot be a folded side of τ .

Let us illustrate Theorem 3.6 with an example.

Example 3.8. In this example we put tagged triangulations completely aside. Consider the ideal triangulation τ° depicted on the right side of Figure 8. Its associated potential is given by

$$S(\tau^\circ, \mathbf{x}) = \alpha\beta\gamma + a_1b_2b_6 + a_2a_6b_1 + a_3b_4a_5 + a_4b_5b_3 + x\delta\varepsilon b_1b_2b_3b_4b_5b_6 - y^{-1}\alpha\delta\varepsilon + z\alpha a_1a_2a_3a_4a_5a_6$$

(note that the terms $x\delta\varepsilon b_1b_2b_3b_4b_5b_6$ and $z\alpha a_1a_2a_3a_4a_5a_6$ are accompanied by a positive sign, whereas $y^{-1}\alpha\delta\varepsilon$ is accompanied by a negative sign; this is in sync with the fact that ideal triangulations correspond to tagged triangulations with non-negative signature).

If we apply the so-called *premutation* $\tilde{\mu}_{k^\circ}$ to $(Q(\tau^\circ), S(\tau^\circ, \mathbf{x}))$ (cf. [10, Equations (5.8) and (5.9)]), we obtain the QP $(\tilde{\mu}_{k^\circ}(Q(\tau^\circ)), \tilde{\mu}_{k^\circ}(S(\tau^\circ, \mathbf{x})))$, where

$$\begin{aligned} \tilde{\mu}_{k^\circ}(S(\tau^\circ, \mathbf{x})) &= [S(\tau^\circ, \mathbf{x})] + \triangle_{k^\circ}(Q(\tau^\circ)) = \beta[\gamma\alpha] + a_1b_2b_6 + a_2[a_6b_1] + a_3b_4a_5 + a_4b_5b_3 \\ &\quad + x\delta[\varepsilon b_1]b_2b_3b_4b_5b_6 - y^{-1}\delta[\varepsilon\alpha] + z a_1a_2a_3a_4a_5[a_6\alpha] \\ &\quad + \gamma^*[\gamma\alpha]\alpha^* + \varepsilon^*[\varepsilon\alpha]\alpha^* + a_6^*[a_6\alpha]\alpha^* + \gamma^*[\gamma b_1]b_1^* + \varepsilon^*[\varepsilon b_1]b_1^* + a_6^*[a_6b_1]b_1^* \end{aligned}$$

(we have not drawn the quiver $\tilde{\mu}_{k^\circ}(Q(\tau^\circ))$). If we then apply Derksen-Weyman-Zelevinsky's reduction process we obtain the QP $(\mu_{k^\circ}(Q(\tau^\circ)), \mu_{k^\circ}(S(\tau^\circ, \mathbf{x})))$, where

$$\begin{aligned} \mu_{k^\circ}(S(\tau^\circ, \mathbf{x})) &= a_1b_2b_6 + a_3b_4a_5 + a_4b_5b_6 + xy\alpha^*\varepsilon^*[\varepsilon b_1]b_2b_3b_4b_5b_6 - zb_1^*a_6^*a_3a_4a_5[a_6\alpha]a_1 \\ &\quad + a_6^*[a_6\alpha]\alpha^* + \gamma^*[\gamma b_1]b_1^* + \varepsilon^*[\varepsilon b_1]b_1^*. \end{aligned}$$

If we on the other hand flip the arc $k^\circ \in \tau^\circ$ we obtain the ideal triangulation $\sigma^\circ = f_{k^\circ}(\tau^\circ)$ depicted on the right side of Figure 9. Its associated potential is

$$S(\sigma^\circ, \mathbf{x}) = \gamma^*[\gamma b_1]b_1^* + a_6^*[a_6\alpha]\alpha^* + a_1b_2b_6 + a_3b_4a_5 + a_4b_5b_3 + x\alpha^*\varepsilon^*[\varepsilon b_1]b_2b_3b_4b_5b_6 - y^{-1}\varepsilon^*[\varepsilon b_1]b_1^* + za_1b_1^*a_6^*a_3a_4a_5[a_6\alpha].$$

The right-equivalence $\varphi : \mu_{k^\circ}(Q(\tau^\circ), S(\tau^\circ, \mathbf{x})) \rightarrow (Q(\sigma^\circ), S(\sigma^\circ, \mathbf{x}))$ witnessing the proof of Theorem 3.6 can be chosen to act on arrows by

$$\varphi : \alpha^* \mapsto -\alpha^*, \varepsilon^* \mapsto -y^{-1}\varepsilon^*, a_6^* \mapsto -a_6^*.$$

We end the example with a remark: The term “premutation”, with which we refer to $\tilde{\mu}_{k^\circ}$, is very rarely used in the literature (in particular, it is not used in [10]). \blacktriangle

4. FLIP/MUTATION OF FOLDED SIDES: THE PROBLEM

Consider the tagged triangulations τ and $\sigma = f_i(\tau)$ of the three-times-punctured hexagon shown in Figure 10, where the quivers $Q(\tau)$ and $Q(\sigma)$ are drawn as well. The signature δ_τ of τ is certainly non-negative,

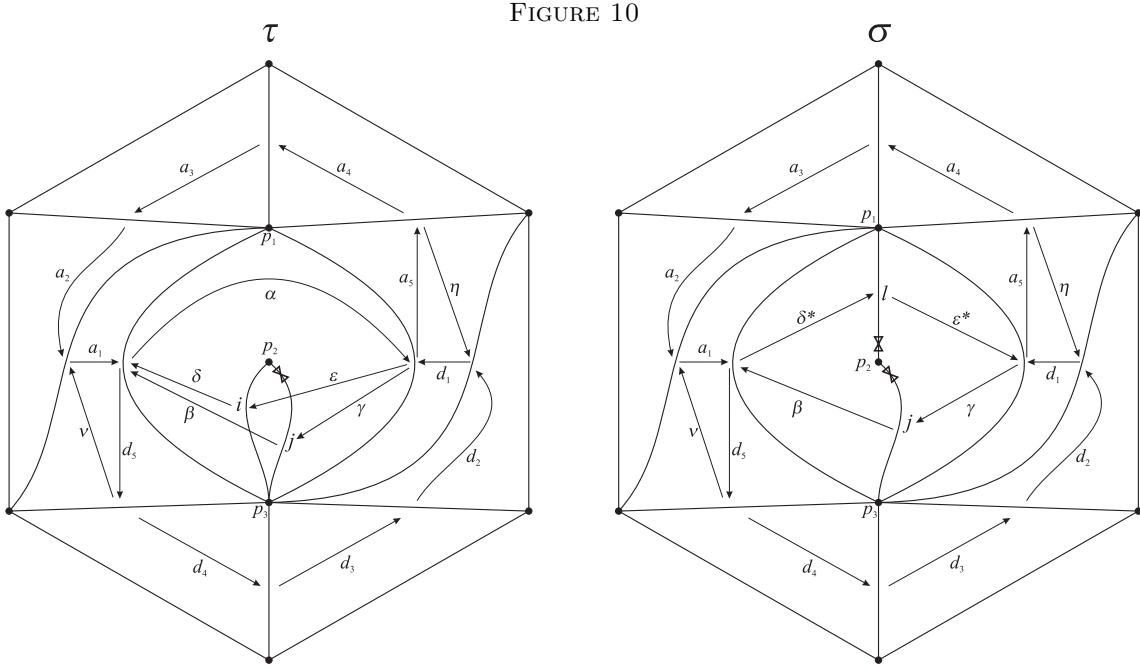


FIGURE 10

which means that τ can be thought to be an ideal triangulation. The arc $i \in \tau$ is then the folded side of a self-folded triangle Δ of τ and the tagged arc j corresponds to the loop contained in Δ that encloses i in τ .

Since σ is obtained from τ by the flip of i , we would “like” the QPs $\mu_i((Q(\tau), S(\tau, \mathbf{x})))$ and $(Q(\sigma), S(\sigma, \mathbf{x}))$ to be right-equivalent, a fact that does not follow from Theorem 3.6, because i is a folded side of τ (see Remark 3.7). According to Definition 3.2, we have

$$\begin{aligned} S(\tau, \mathbf{x}) &= \alpha\beta\gamma + a_1\nu d_5 + a_5d_1\eta + x_{p_1}\alpha a_1a_2a_3a_4a_5 - x_{p_2}^{-1}\alpha\delta\varepsilon + x_{p_3}\delta\varepsilon d_1d_2d_3d_4d_5 \quad \text{and} \\ S(\sigma, \mathbf{x}) &= a_1\nu d_5 + a_5d_1\eta + x_{p_1}\varepsilon^*\delta^*a_1a_2a_3a_4a_5 + x_{p_2}^{-1}\varepsilon^*\delta^*\beta\gamma + x_{p_3}\beta\gamma d_1d_2d_3d_4d_5. \end{aligned}$$

If we apply the i^{th} QP-mutation to $(Q(\tau), S(\tau, \mathbf{x}))$ we obtain the QP $(\mu_i(Q(\tau)), \mu_i(S(\tau, \mathbf{x})))$, where $\mu_i(Q(\tau)) = Q(\sigma)$ and

$$\begin{aligned} \mu_i(S(\tau, \mathbf{x})) &= a_1\nu d_5 + a_5d_1\eta + x_{p_1}x_{p_2}\varepsilon^*\delta^*a_1a_2a_3a_4a_5 + x_{p_2}\varepsilon^*\delta^*\beta\gamma + x_{p_2}x_{p_3}\beta\gamma d_1d_2d_3d_4d_5 \\ &\quad + x_{p_1}x_{p_2}x_{p_3}d_1d_2d_3d_4d_5a_1a_2a_3a_4a_5. \end{aligned}$$

After a quick look at $S(\sigma, \mathbf{x})$ and $\mu_i(S(\tau, \mathbf{x}))$ we see that not only do the scalars accompanying the cycles around the punctures not match, the set of (rotation classes of) cycles appearing in $S(\sigma, \mathbf{x})$ and $\mu_i(S(\tau, \mathbf{x}))$ do not match either: the cycle $d_1d_2d_3d_4d_5a_1a_2a_3a_4a_5$ does not appear in $S(\sigma, \mathbf{x})$.

The appearance of the aforementioned cycle in $\mu_i(S(\tau, \mathbf{x}))$ is general: whenever we have a self-folded triangle in an ideal triangulation τ , with the property that the digon surrounding the self-folded triangle is entirely contained in the interior of (Σ, \mathbb{M}) (so that all marked points appearing in the digon are punctures), the QP-mutation with respect to the folded side will involve all the cycles appearing in $S(\sigma, \mathbf{x})$ and the cycle surrounding the digon. This may suggest that our definition of $S(\sigma, \mathbf{x})$ is “wrong” and “should” be changed taking the alluded cycle into account. However, if we make such change, after a few flips/mutations we will soon lose control over the potentials, for “many” “strange” cycles will have to be taken into account in each potential (not only “many” at a time, also “stranger and stranger”).

There is nevertheless another possibility: the definition of $S(\sigma, \mathbf{x})$ may indeed be “the right one”. If so, then we have to show that there is an R -algebra isomorphism $\varphi : R\langle\langle\mu_i(Q(\tau))\rangle\rangle \rightarrow R\langle\langle Q(\sigma)\rangle\rangle$ that simultaneously eliminates the digon-surrounding cycle and makes the coefficients of puncture-surrounding cycles match.

In the very particular situation of Figure 10, the R -algebra isomorphism $\varphi : R\langle\langle Q(\sigma)\rangle\rangle \rightarrow R\langle\langle\mu_i(Q(\tau))\rangle\rangle$ given by

$$(4.1) \quad \nu \mapsto \nu + x_{p_1}x_{p_2}x_{p_3}a_2a_3a_4a_5d_1d_2d_3d_4, \quad \varepsilon^* \mapsto x_{p_2}\varepsilon^*, \quad \beta \mapsto x_{p_2}\beta,$$

and the identity on the remaining arrows of $Q(\sigma)$, is a right-equivalence $(Q(\sigma), S(\sigma, \mathbf{x})) \rightarrow (\mu_i(Q(\tau)), \mu_i(S(\tau, \mathbf{x})))$ (and so, φ^{-1} eliminates the digon-surrounding cycle and makes the coefficients match). But if we have a careful look at the potentials $S(\sigma, \mathbf{x})$ and $\mu_i(S(\tau, \mathbf{x}))$, we realize that the reason why φ serves as a right-equivalence in this particular situation is the combination of the following two facts:

- The arrow ν appears in only one term of $S(\sigma, \mathbf{x})$;
- $\partial_\nu(S(\sigma, \mathbf{x}))$ is a factor of the cycle in $\mu_i(S(\tau, \mathbf{x}))$ that does not appear in $S(\sigma, \mathbf{x})$.

Had ν appeared in two terms of $S(\sigma, \mathbf{x})$, then the rule (4.1) would have added two terms to $S(\sigma, \mathbf{x})$, not only one, and would have failed to be a right-equivalence $(Q(\sigma), S(\sigma, \mathbf{x})) \rightarrow (\mu_i(Q(\tau)), \mu_i(S(\tau, \mathbf{x})))$.

Now, the reason why ν appears in only one term of $S(\sigma, \mathbf{x})$ is the fact that it arises from the (signed) adjacency between two arcs at a marked point lying on $\partial\Sigma$. All in all, this means that for (4.1) to indeed define a desired right-equivalence, we ultimately relied on the fact that the underlying surface has non-empty boundary. We want to show that the desired right-equivalence exists independently of any assumption on the boundary.

5. POPPED POTENTIALS: DEFINITION AND FLIP/MUTATION DYNAMICS

Here we go back to formal considerations. Let (Σ, \mathbb{M}) be a surface. Throughout this section we do not impose any assumption on the boundary of Σ nor on the number of punctures.

Suppose i is the folded side of a self-folded triangle of an ideal triangulation τ . Let $j \in \tau$ be the loop that cuts out a once-punctured monogon and encloses i , and $q \in \mathbb{P}$ be the puncture that lies inside the monogon cut out by j . Let $\pi_{i,j}^\tau : \tau \rightarrow \tau$ be the bijection that fixes $\tau \setminus \{i, j\}$ pointwise and interchanges i and j . Then $\pi_{i,j}^\tau$ extends uniquely to an automorphism of the quiver $Q(\tau)$, and therefore, to a K -algebra automorphism of $R\langle\langle Q(\tau)\rangle\rangle$. In a slight notational abuse we shall denote this quiver and algebra automorphisms by $\pi_{i,j}^\tau$ as well.

Remark 5.1. Notice that $\pi_{i,j}^\tau$ is not an R -algebra automorphism of $R\langle\langle Q(\tau)\rangle\rangle$.

Definition 5.2. Let $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ be a choice of non-zero scalars. With the notations from the preceding paragraph, let $S(\tau, \mathbf{y})$ be the potential associated to τ with respect to the choice of scalars $\mathbf{y} = (y_p)_{p \in \mathbb{P}}$ defined by $y_p = (-1)^{\delta_{p,q}}x_p$ for all $p \in \mathbb{P}$, where $\delta_{p,q}$ is the Kronecker delta. The *popped potential* of the quadruple (τ, \mathbf{x}, i, j) is the potential $W(\tau, \mathbf{x}) = \pi_{i,j}^\tau(S(\tau, \mathbf{y})) \in R\langle\langle Q(\tau)\rangle\rangle$.

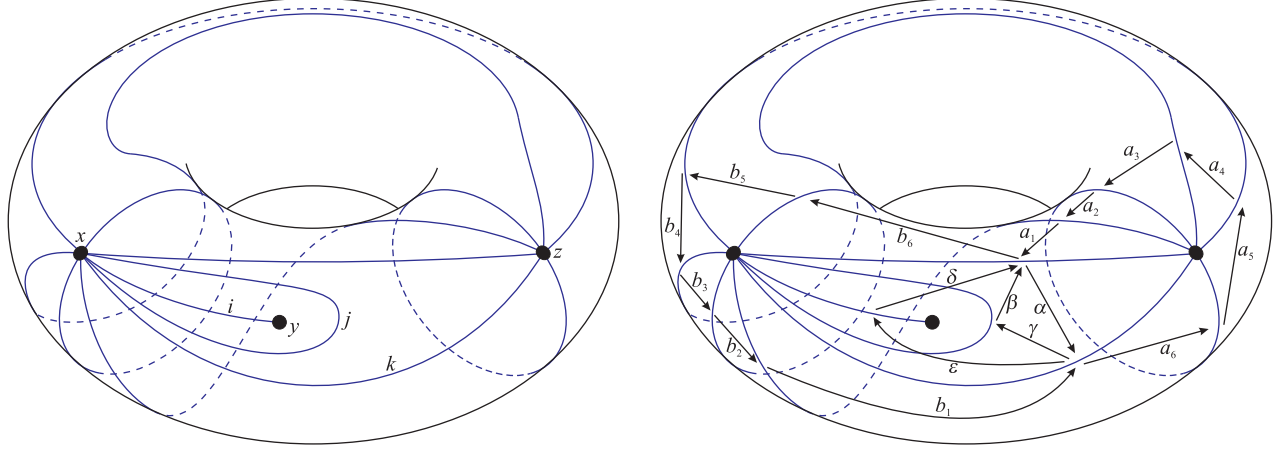
Remark 5.3. (1) The K -algebra automorphism $\pi_{i,j}^\tau$ of $R\langle\langle Q(\tau)\rangle\rangle$ has only been used to define $W(\tau, \mathbf{x})$, and it is actually possible to define $W(\tau, \mathbf{x})$ without any mention of $\pi_{i,j}^\tau$ (we have not done so in order to avoid an unnecessarily cumbersome definition), but we have not claimed that $\pi_{i,j}^\tau$ is a right-equivalence: it cannot be, precisely because it does not act as the identity on the vertex span R (see [10, Definition 4.2]).

(2) Note that an ideal triangulation τ can give rise to several popped potentials: each self-folded triangle of τ gives rise to one such. So, when speaking of the popped potential associated to an

ideal triangulation, one has to specify the self-folded triangle with respect to which one is taking such potential.

Example 5.4. On the left side of Figure 11 we can see a three-times-punctured torus (with no boundary), with the scalars x_p , $p \in \mathbb{P}$, labeling the three punctures, and an ideal triangulation τ of it that has a (unique) self-folded triangle. On the right side we can see the quiver $Q(\tau)$ drawn on the surface. Definition

FIGURE 11



3.2 tells us that

$$S(\tau, \mathbf{x}) = \alpha\beta\gamma + a_1b_2b_6 + a_2a_6b_1 + a_3b_4a_5 + a_4b_5b_3 + x\delta\varepsilon b_1b_2b_3b_4b_5b_6 - y^{-1}\alpha\delta\varepsilon + z\alpha a_1a_2a_3a_4a_5a_6,$$

whereas Definition 5.2 tells us that the popped potential is

$$W(\tau, \mathbf{x}) = \alpha\delta\varepsilon + a_1b_2b_6 + a_2a_6b_1 + a_3b_4a_5 + a_4b_5b_3 + x\beta\gamma b_1b_2b_3b_4b_5b_6 + y^{-1}\alpha\beta\gamma + z\alpha a_1a_2a_3a_4a_5a_6.$$

▲

Lemma 5.5. *Let τ an ideal triangulation, and $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ and $i, j \in \tau$ be as above. If $k \in \tau \setminus \{i, j\}$ is an arc such that $\sigma = f_k(\tau)$ happens to be an ideal triangulation, then $\mu_k(Q(\tau), W(\tau, \mathbf{x}))$ is right-equivalent to $(Q(\sigma), W(\sigma, \mathbf{x}))$, where $W(\sigma, \mathbf{x})$ is the popped potential of the quadruple $(\sigma, \mathbf{x}, i, j)$.*

Proof. Let $\mathbf{y} = ((-1)^{\delta_{p,q}} x_p)_{p \in \mathbb{P}}$ be as in Definition 5.2. Notice that the bijection $\pi = \pi_{i,j}^\tau : \tau \rightarrow \tau$ induces a quiver automorphism ψ of $\tilde{\mu}_k(Q(\tau))$. Following a slight notational abuse, we shall denote also by ψ the induced K -algebra automorphism of $R\langle\langle\tilde{\mu}_k(Q(\tau))\rangle\rangle$ (which is not an R -algebra automorphism). It is straightforward to see that $\psi([S(\tau, \mathbf{y})]) = [\pi(S(\tau, \mathbf{y}))]$ and $\psi(\Delta_k(Q)) = \Delta_k(Q)$, and hence

$$\begin{aligned} \psi(\tilde{\mu}_k(S(\tau, \mathbf{y}))) &= \psi([S(\tau, \mathbf{y})] + \Delta_k(Q(\tau))) && \text{by [10, Equations (5.8) and (5.9)]} \\ &= [\pi(S(\tau, \mathbf{y}))] + \Delta_k(Q(\tau)) \\ &= \tilde{\mu}_k(\pi(S(\tau, \mathbf{y}))) && \text{by [10, Equations (5.8) and (5.9)]} \\ &= \tilde{\mu}_k(W(\tau, \mathbf{x})) && \text{by Definition 5.2.} \end{aligned}$$

Since $\sigma = f_k(\tau)$ is an ideal triangulation, Theorem 3.6 guarantees the existence of a right-equivalence $\varphi : \tilde{\mu}_k(Q(\tau), S(\tau, \mathbf{y})) \rightarrow (Q(\sigma), S(\sigma, \mathbf{y})) \oplus (C, T)$, where (C, T) is a trivial QP. This, together with the obvious fact that $\psi\varphi\psi^{-1}$ acts as the identity on the vertex span R , implies that $\psi\varphi\psi^{-1} : R\langle\langle\tilde{\mu}_k(Q(\tau))\rangle\rangle \rightarrow R\langle\langle Q(\sigma) \oplus C \rangle\rangle$ is an R -algebra isomorphism such that $\psi\varphi\psi^{-1}(\tilde{\mu}_k(W(\tau, \mathbf{x}))) = \psi\varphi(\tilde{\mu}_k(S(\tau, \mathbf{y})))$ is cyclically-equivalent to $\psi(S(\sigma, \mathbf{y}) + T) = W(\sigma, \mathbf{x}) + \psi(T)$. That is, $\psi\varphi\psi^{-1}$ is a right-equivalence $\psi\varphi\psi^{-1} : \tilde{\mu}_k(Q(\tau), W(\tau, \mathbf{x})) \rightarrow (Q(\sigma), W(\sigma, \mathbf{x})) \oplus (\psi(C), \psi(T))$. Since $(Q(\sigma), W(\sigma, \mathbf{x}))$ is a reduced QP and $(\psi(C), \psi(T))$ is a trivial QP, Derksen-Weyman-Zelevinsky's Splitting Theorem [10, Theorem 4.6] allows us to conclude that the QPs $\mu_k(Q(\tau), W(\tau, \mathbf{x}))$ and $(Q(\sigma), W(\sigma, \mathbf{x}))$ are right-equivalent. \square

Intuitively and roughly speaking, Lemma 5.5 says that for a fixed self-folded triangle Δ , the popped QPs $(Q(\tau), W(\tau, \mathbf{x}))$ associated to ideal triangulations containing Δ have the same QP-mutation dynamics (ie, compatible with flips) as the QPs $(Q(\tau), S(\tau, \mathbf{x}))$ as long as the two arcs in Δ are never flipped.

Example 5.6. Let us illustrate the proof of Lemma 5.5 in the situation of Example 5.4. The quiver automorphism $\pi = \pi_{i,j}^\tau : Q(\tau) \rightarrow Q(\tau)$ is easily seen to act by the rules

$$\pi : \quad \beta \mapsto \delta, \quad \delta \mapsto \beta, \quad \gamma \mapsto \varepsilon, \quad \varepsilon \mapsto \gamma.$$

Although we have not drawn the quiver $\tilde{\mu}_k(Q(\tau))$, this implies that the quiver automorphism $\psi : \tilde{\mu}_k(Q(\tau)) \rightarrow \tilde{\mu}_k(Q(\tau))$ acts by the rules

$$\psi : \quad \beta \mapsto \delta, \quad \delta \mapsto \beta, \quad \gamma^* \mapsto \varepsilon^*, \quad \varepsilon^* \mapsto \gamma^*, \quad [\gamma\alpha] \mapsto [\varepsilon\alpha], \quad [\varepsilon\alpha] \mapsto [\gamma\alpha], \quad [\gamma b_1] \mapsto [\varepsilon b_1], \quad [\varepsilon b_1] \mapsto [\gamma b_1],$$

and hence

$$\begin{aligned} \psi([S(\tau, \mathbf{y})]) &= \psi(\beta[\gamma\alpha] + a_1 b_2 b_6 + a_2 [a_6 b_1] + a_3 b_4 a_5 + a_4 b_5 b_3 + x\delta[\varepsilon b_1] b_2 b_3 b_4 b_5 b_6 + y^{-1}\delta[\varepsilon\alpha] + z a_1 a_2 a_3 a_4 a_5 [a_6\alpha]) \\ &= \delta[\varepsilon\alpha] + a_1 b_2 b_6 + a_2 [a_6 b_1] + a_3 b_4 a_5 + a_4 b_5 b_3 + x\beta[\gamma b_1] b_2 b_3 b_4 b_5 b_6 + y^{-1}\beta[\gamma\alpha] + z a_1 a_2 a_3 a_4 a_5 [a_6\alpha]. \end{aligned}$$

Since

$$\begin{aligned} [\pi(S(\tau, \mathbf{y}))] &= [\alpha\delta\varepsilon + a_1 b_2 b_6 + a_2 a_6 b_1 + a_3 b_4 a_5 + a_4 b_5 b_3 + x\beta\gamma b_1 b_2 b_3 b_4 b_5 b_6 + y^{-1}\alpha\beta\gamma + z\alpha a_1 a_2 a_3 a_4 a_5 a_6] \\ &= \delta[\varepsilon\alpha] + a_1 b_2 b_6 + a_2 [a_6 b_1] + a_3 b_4 a_5 + a_4 b_5 b_3 + x\beta[\gamma b_1] b_2 b_3 b_4 b_5 b_6 + y^{-1}\beta[\gamma\alpha] + z a_1 a_2 a_3 a_4 a_5 [a_6\alpha], \end{aligned}$$

the equality $\psi([S(\tau, \mathbf{y})]) = [\pi(S(\tau, \mathbf{y}))]$ is obvious.

Turning to $\Delta_k(Q(\tau))$ and $\psi(\Delta_k(Q(\tau)))$, we have

$$\Delta_k(Q(\tau)) = \gamma^*[\gamma\alpha]\alpha^* + \varepsilon^*[\varepsilon\alpha]\alpha^* + a_6^*[a_6\alpha]\alpha^* + \gamma^*[\gamma b_1]b_1^* + \varepsilon^*[\varepsilon b_1]b_1^* + a_6^*[a_6 b_1]b_1^*$$

and hence

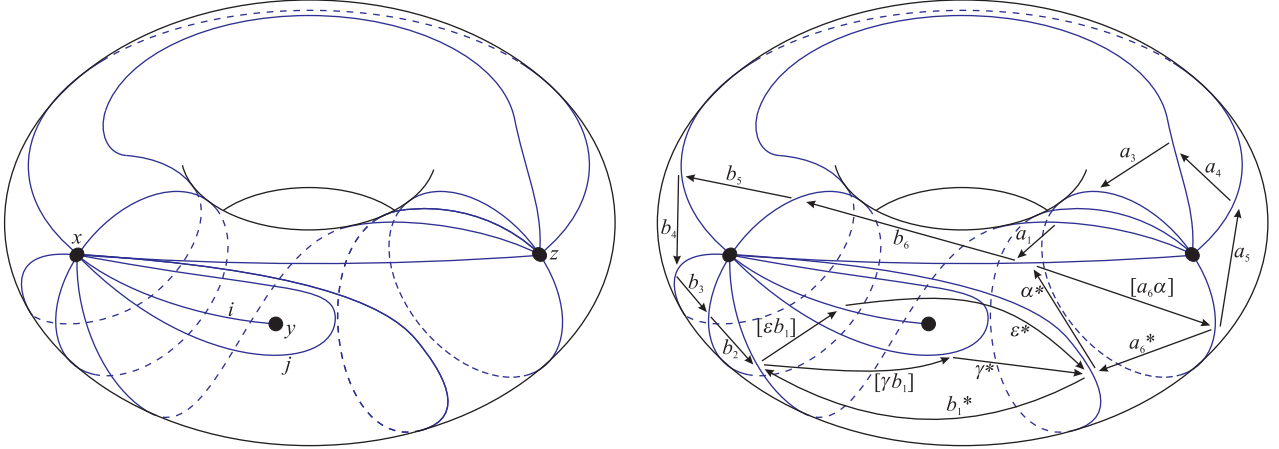
$$\psi(\Delta_k(Q(\tau))) = \varepsilon^*[\varepsilon\alpha]\alpha^* + \gamma^*[\gamma\alpha]\alpha^* + a_6^*[a_6\alpha]\alpha^* + \varepsilon^*[\varepsilon b_1]b_1^* + \gamma^*[\gamma b_1]b_1^* + a_6^*[a_6 b_1]b_1^*$$

obviously equals $\Delta_k(Q(\tau))$.

Therefore, as stated in the proof of Lemma 5.5, we have $\psi(\tilde{\mu}_k(S(\tau, \mathbf{y}))) = \psi([S(\tau, \mathbf{y})] + \Delta_k(Q(\tau))) = [\pi(S(\tau, \mathbf{y}))] + \Delta_k(Q(\tau)) = \tilde{\mu}_k(\pi(S(\tau, \mathbf{y}))) = \tilde{\mu}_k(W(\tau, \mathbf{x}))$.

On the other hand, in Figure 12 we see the ideal triangulation $\sigma = f_k(\tau)$ that results from flipping the arc k of the ideal triangulation τ shown in Figure 11.

FIGURE 12. $\sigma = f_k(\tau)$ (cf. Figure 11)



Since it is the scalars x_p , $p \in \mathbb{P}$, that are labeling the punctures, the potential associated to σ with respect to the choice $\mathbf{y} = ((-1)^{\delta_{p,q}} x_p)_{p \in \mathbb{P}}$ is

$$\begin{aligned} S(\sigma, \mathbf{y}) &= \gamma^*[\gamma b_1]b_1^* + a_6^*[a_6\alpha]\alpha^* + a_1 b_2 b_6 + a_3 b_4 a_5 + a_4 b_5 b_3 + x[\varepsilon b_1] b_2 b_3 b_4 b_5 b_6 \alpha^* \varepsilon^* + y^{-1} \varepsilon^*[\varepsilon b_1]b_1^* \\ &\quad + z a_1 a_2 a_3 a_4 a_5 [a_6\alpha], \end{aligned}$$

hence the corresponding popped potential is

$$\begin{aligned} S(\sigma, \mathbf{x}) &= \varepsilon^*[\varepsilon b_1]b_1^* + a_6^*[a_6\alpha]\alpha^* + a_1 b_2 b_6 + a_3 b_4 a_5 + a_4 b_5 b_3 + x[\gamma b_1] b_2 b_3 b_4 b_5 b_6 \alpha^* \gamma^* + y^{-1} \gamma^*[\gamma b_1]b_1^* \\ &\quad + z a_1 a_2 a_3 a_4 a_5 [a_6\alpha]. \end{aligned}$$

The quiver $C = (Q_0, C_1, t, h)$, the potential T and the right-equivalence $\varphi : \tilde{\mu}_k(Q(\tau), S(\tau, \mathbf{y})) \rightarrow (Q(\sigma), S(\sigma, \mathbf{y})) \oplus (C, T)$ can be chosen so that

$$\begin{aligned} C_1 &= \{[\gamma\alpha], \beta, [\varepsilon\alpha], \delta, [a_6b_1], a_2\}, \text{ with the functions } t, h : C_1 \rightarrow Q_0 \text{ defined in the obvious way,} \\ T &= \beta[\gamma\alpha] + a_2[a_6a_1] + y^{-1}\delta[\varepsilon\alpha], \\ \varphi : \quad &\beta \mapsto \beta + \alpha^*\gamma^*, \quad a_2 \mapsto a_2 - b_1^*a_6^*, \quad [a_6b_1] \mapsto [a_6b_1] + za_3a_4a_5[a_6\alpha]a_1, \quad \delta \mapsto \delta + \alpha^*\varepsilon^*, \\ &[\varepsilon\alpha] \mapsto [\varepsilon\alpha] - xy[\varepsilon b_1]b_2b_3b_4b_5b_6, \quad \varepsilon^* \mapsto y^{-1}\varepsilon^*, \quad \alpha^* \mapsto -\alpha^*, \quad [a_6\alpha] \mapsto -[a_6\alpha]. \end{aligned}$$

This means that the right-equivalence $\psi\varphi\psi^{-1} : \tilde{\mu}_k(Q(\tau), W(\tau, \mathbf{x})) \rightarrow (Q(\sigma), W(\sigma, \mathbf{x})) \oplus (\psi(C), \psi(T))$ acts according to the rules

$$\begin{aligned} \psi\varphi\psi^{-1} : \quad &\delta \mapsto \delta + \alpha^*\varepsilon^*, \quad a_2 \mapsto a_2 - b_1^*a_6^*, \quad [a_6b_1] \mapsto [a_6b_1] + za_3a_4a_5[a_6\alpha]a_1, \quad \beta \mapsto \beta + \alpha^*\gamma^*, \\ &[\gamma\alpha] \mapsto [\gamma\alpha] - xy[\gamma b_1]b_2b_3b_4b_5b_6, \quad \gamma^* \mapsto y^{-1}\gamma^*, \quad \alpha^* \mapsto -\alpha^*, \quad [a_6\alpha] \mapsto -[a_6\alpha]. \end{aligned}$$

In other words, the desired right-equivalence between $\mu_k(Q(\tau), W(\tau, \mathbf{x}))$ and $(Q(\sigma), W(\sigma, \mathbf{x}))$ can be obtained from the existent one between $\mu_k(Q(\tau), S(\tau, \mathbf{y}))$ and $(Q(\sigma), S(\sigma, \mathbf{y}))$ by formally applying the symbol exchanges $\beta \leftrightarrow \delta$ and $\gamma \leftrightarrow \varepsilon$. \blacktriangle

6. POP IS RIGHT-EQUIVALENCE: STATEMENT AND PROOF

Let (Σ, \mathbb{M}) be a surface. We assume that

$$(6.1) \quad (\Sigma, \mathbb{M}) \text{ is not a sphere with less than 6 punctures.}$$

This is actually an extremely mild assumption. Indeed, all the following surfaces satisfy (6.1):

$$(6.2) \quad \text{all surfaces with non-empty boundary, with or without punctures, regardless of their genus;}$$

$$(6.3) \quad \text{all positive-genus surfaces without boundary, and any number of punctures;}$$

$$(6.4) \quad \text{all spheres with at least 6 punctures.}$$

6.1. Statement. The main result of the section (and instrumentally the main result of the paper) is the Popping Theorem, which we now state.

Theorem 6.1 (Popping Theorem). *Suppose that (Σ, \mathbb{M}) satisfies (6.1). Let σ be any ideal triangulation of (Σ, \mathbb{M}) . If $i \in \sigma$ is a folded side of σ , enclosed by the loop j , then $(Q(\sigma), S(\sigma, \mathbf{x}))$ is right-equivalent to $(Q(\sigma), W(\sigma, \mathbf{x}))$, where $W(\sigma, \mathbf{x})$ is the popped potential of the quadruple $(\sigma, \mathbf{x}, i, j)$.*

Proof. Suppose for a moment that there exists an ideal triangulation τ with $i, j \in \tau$, such that the pop of (i, j) in τ induces right-equivalence, that is, such that

$$(6.5) \quad (Q(\tau), S(\tau, \mathbf{x})) \text{ is right-equivalent to } (Q(\tau), W(\tau, \mathbf{x})),$$

where $(Q(\tau), W(\tau, \mathbf{x}))$ is the popped potential of the quadruple (τ, \mathbf{x}, i, j) . By 2.7, there exists an ideal flip-sequence $(\sigma_0, \sigma_1, \dots, \sigma_s)$ with $\sigma_0 = \sigma$ and $\sigma_s = \tau$, such that $i, j \in \sigma_r$ for all $r = 0, \dots, s$. Let $k_1 \in \sigma_1, k_2 \in \sigma_2, \dots, k_s \in \sigma_s$ be the (ordinary) arcs such that $\sigma_{r-1} = f_{k_r}(\sigma_r)$ for $r = 1, \dots, s$. Then, using the symbol \simeq to abbreviate “is right-equivalent to”,

$$\begin{aligned} (Q(\sigma), S(\sigma, \mathbf{x})) &\simeq \mu_{k_1}\mu_{k_2}\dots\mu_{k_s}(Q(\tau), S(\tau, \mathbf{x})) && \text{(by Theorem 3.6)} \\ &\simeq \mu_{k_1}\mu_{k_2}\dots\mu_{k_s}(Q(\tau), W(\tau, \mathbf{x})) && \text{(by (6.5) and [10, Corollary 5.4])} \\ &\simeq (Q(\sigma), W(\sigma, \mathbf{x})) && \text{(by Lemma 5.5).} \end{aligned}$$

Theorem 6.1 is thus proved up to showing the existence of an ideal triangulation τ such that $i, j \in \tau$ and satisfying (6.5). \square

We devote the rest of the section to provide the piece missing in the proof of the Popping Theorem 6.1, namely:

Proposition 6.2. *Suppose (Σ, \mathbb{M}) is not a sphere with less than 6 punctures. Let i and j be (ordinary) arcs on (Σ, \mathbb{M}) , such that j is a loop cutting out a once-punctured monogon, and i is the unique arc that connects the basepoint of j with the puncture enclosed by j and is entirely contained in the once-punctured monogon cut out by j . Then there exists an ideal triangulation τ of (Σ, \mathbb{M}) such that $i, j \in \tau$ and satisfying (6.5).*

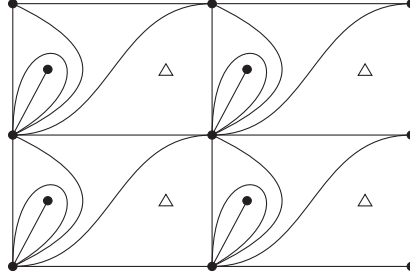
6.2. Proof for positive-genus surfaces with empty boundary. In the case of surfaces with empty boundary, all marked points have the same nature (this can be stated formally by saying that any permutation of the puncture set \mathbb{P} can be extended to an orientation-preserving auto-diffeomorphism of Σ). Thus, in order to prove Proposition 6.2 and this way finish the proof of the Popping Theorem for empty-boundary surfaces, it suffices to show the existence of an ideal triangulation τ having a self-folded triangle whose pop in τ induces right-equivalence. In this subsection we exhibit such a τ under the assumption that the underlying surface has empty boundary and positive genus.

We start by constructing a τ with the desired property in the case where (Σ, \mathbb{M}) is not a twice- or three-times-punctured torus. Let $(\mathbb{T}, \{p_1, p_2\})$ be a torus with empty boundary and exactly two punctures. Also, let Σ' be a compact connected oriented Riemann surface with exactly one boundary component (we allow Σ' to have genus 0, that is, we allow it to be a disc), and let $\mathbb{M}' \subset \Sigma'$ be a possibly empty finite set of points on Σ' , with the following two properties:

- No element of \mathbb{M}' lies on the boundary of Σ' ;
- if Σ' is a disk, then $|\mathbb{M}'| \geq 2$.

Let σ be the ideal triangulation of $(\mathbb{T}, \{p_1, p_2\})$ sketched in Figure 13. Inside the triangle Δ signaled in

FIGURE 13. Twice-punctured torus with empty boundary



that same Figure, cut out an open disc D (so that $\mathbb{T} \setminus D$ is a compact torus with exactly one boundary component) and glue $\mathbb{T} \setminus D$ with Σ' along boundary components, so that the resulting surface Σ is the well-known connected sum $\mathbb{T} \# (\Sigma')$. Setting $\mathbb{M} = \{p_1, p_2\} \cup \mathbb{M}'$, we see that

- (Σ, \mathbb{M}) is a positive-genus surface with empty boundary and at least two punctures;
- (Σ, \mathbb{M}) is not a torus with exactly two or three punctures;
- σ is a set of pairwise compatible arcs on (Σ, \mathbb{M}) .

Complete σ to an ideal triangulation τ of (Σ, \mathbb{M}) , in such a way that τ has exactly one self-folded triangle, see Figure 14. With the notation of that same Figure, we see that the potentials $S(\tau, \mathbf{x})$ and $W(\tau, \mathbf{x})$ are

$$(6.6) \quad S(\tau, \mathbf{x}) = -x_{p_2}^{-1} \alpha \delta \varepsilon + \eta_1 \eta_2 \eta_3 + \lambda_1 \lambda_2 \lambda_3 + \alpha \beta \gamma + x_{p_1} \alpha \nu_2 \lambda_3 \eta_3 \lambda_2 \nu_3 \rho \nu_1 \delta \varepsilon \eta_2 \lambda_1 \eta_1 + S'(\tau),$$

$$(6.7) \quad W(\tau, \mathbf{x}) = \alpha \delta \varepsilon + \eta_1 \eta_2 \eta_3 + \lambda_1 \lambda_2 \lambda_3 + x_{p_2}^{-1} \alpha \beta \gamma + x_{p_1} \alpha \nu_2 \lambda_3 \eta_3 \lambda_2 \nu_3 \rho \nu_1 \beta \gamma \eta_2 \lambda_1 \eta_1 + S'(\tau),$$

where $S'(\tau)$ is a potential not involving any of the arrows $\alpha, \beta, \gamma, \delta, \varepsilon, \eta_1, \eta_2, \eta_3, \lambda_1, \lambda_2, \lambda_3$.

We still have to define τ in the cases where (Σ, \mathbb{M}) is a torus with two or three punctures. For the twice-punctured torus, we take τ to be the triangulation depicted in Figure 15. Using the notation of this Figure, and defining ν_1, ν_2, ν_3 and ρ to be all equal to the idempotent $e_{t(\alpha)} = e_{h(\delta)} = e_{t(\lambda_2)} = e_{h(\lambda_3)}$, we see that the potentials $S(\tau, \mathbf{x})$ and $W(\tau, \mathbf{x})$ are given precisely by the formulas (6.6) and (6.7).

Finally, for the torus with exactly three punctures, we take τ to be the triangulation depicted in Figure 16. Using the notation of this Figure, and defining ρ to be the idempotent $e_{h(\nu_1)} = e_{t(\nu_3)}$, we see that the potentials $S(\tau, \mathbf{x})$ and $W(\tau, \mathbf{x})$ are given precisely by the formulas (6.6) and (6.7) (note that, even though τ has two self-folded triangles, we are popping only one of them, namely, the one containing the puncture p_2 ; that is, $W(\tau, \mathbf{x})$ is the popped potential of τ with respect to the self-folded triangle containing p_2 , see Remark 5.3).

Now that τ has been defined for any empty-boundary surface with at least two punctures, we are ready to state:

Proposition 6.3. *For the ideal triangulation τ just defined, the QPs $(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\tau), W(\tau, \mathbf{x}))$ are right-equivalent.*

FIGURE 14. Positive-genus empty-boundary surface different from a twice- or three-times-punctured torus: Triangulation τ and associated quiver $Q(\tau)$

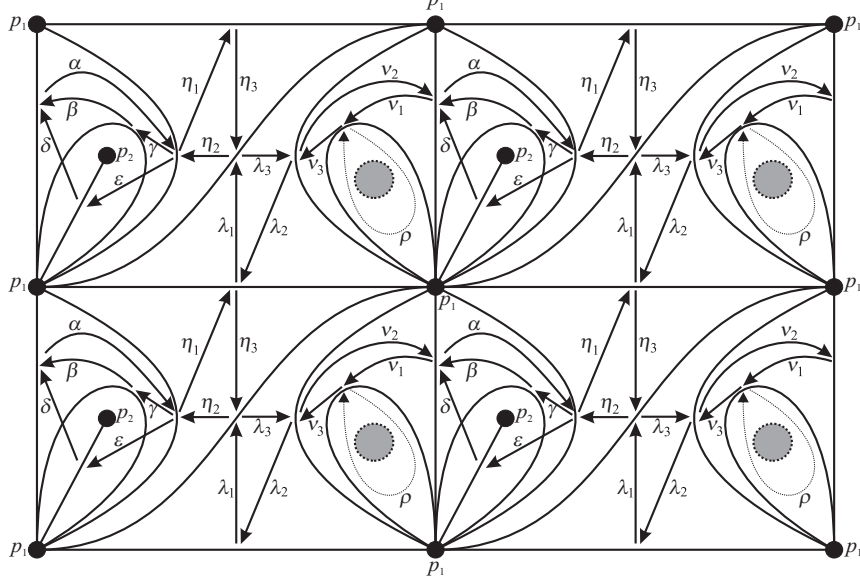
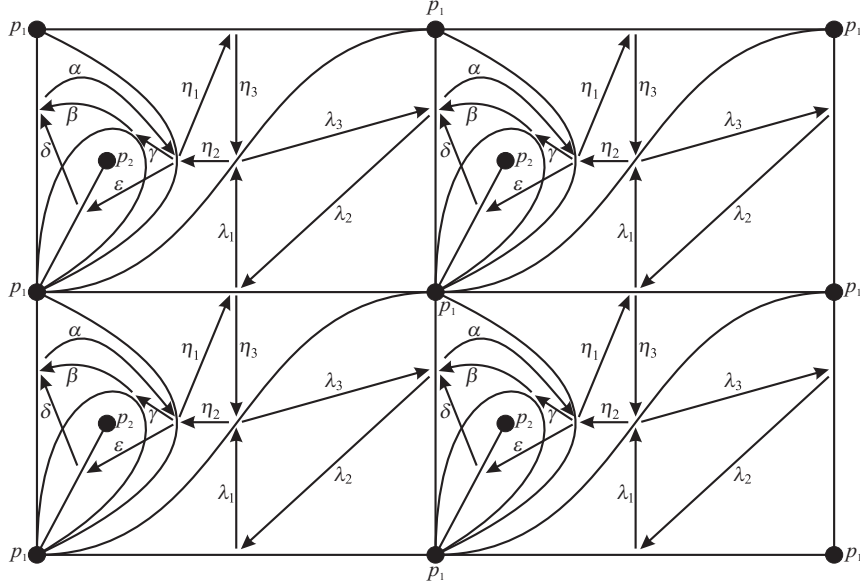


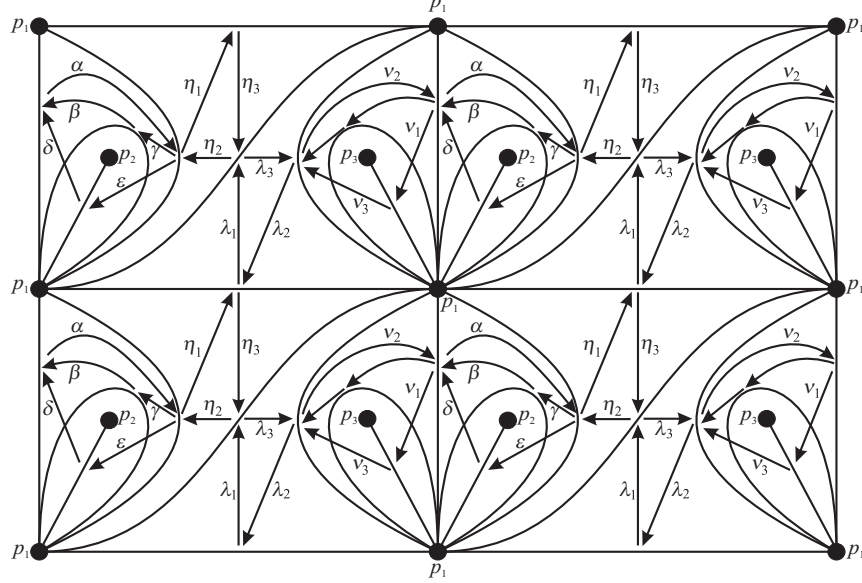
FIGURE 15. Twice-punctured torus with empty boundary: triangulation τ and associated quiver $Q(\tau)$



The proof of Proposition 6.3 will follow from the next two lemmas.

Lemma 6.4. *For the ideal triangulation τ above, the QP $(Q(\tau), S(\tau, \mathbf{x}))$ is right-equivalent to a QP of the form $(Q(\tau), W(\tau, \mathbf{x}) + L_1)$, where $L_1 \in R\langle\langle Q(\tau) \rangle\rangle$ is a potential satisfying the following condition:*

$$(6.8) \quad \text{every cycle appearing in it has at least one of the paths } \eta_2\lambda_1\lambda_2\nu_3 \text{ and } \eta_2\eta_3\lambda_2\nu_3 \text{ as a factor.}$$

FIGURE 16. Three-times-punctured torus with empty boundary: triangulation τ and associated quiver $Q(\tau)$ 

Proof. Write $\Pi = \lambda_1 \eta_1 \alpha \nu_2 \lambda_3 \eta_3 \lambda_2 \nu_3 \rho \nu_1 \nu_2 \lambda_3 \eta_3 \lambda_2 \nu_3 \rho \nu_1 \delta \varepsilon$ and define R -algebra automorphisms $\psi_{\beta, \delta}, \psi_\alpha, \psi_{\eta_3} : R\langle\langle Q(\tau) \rangle\rangle \rightarrow R\langle\langle Q(\tau) \rangle\rangle$ according to the rules

$$\begin{aligned} \psi_{\beta, \delta} &: \beta \mapsto x_{p_2}^{-1} \beta, \quad \delta \mapsto -x_{p_2} \delta, \\ \psi_\alpha &: \alpha \mapsto \alpha + x_{p_1} x_{p_2} \eta_2 \lambda_1 \eta_1 \alpha \nu_2 \lambda_3 \eta_3 \lambda_2 \nu_3 \rho \nu_1 \\ \psi_{\eta_3} &: \eta_3 \mapsto \eta_3 + x_{p_1}^2 x_{p_2}^2 \Pi \eta_2 \lambda_1. \end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned} \psi_{\eta_3} \psi_\alpha \psi_{\beta, \delta}(S(\tau, \mathbf{x})) &\sim_{\text{cyc}} W(\tau, \mathbf{x}) \\ &+ x_{p_1}^3 x_{p_2}^2 \eta_2 \lambda_1 \lambda_2 \nu_3 \rho \nu_1 \beta \gamma \eta_2 \lambda_1 \eta_1 \alpha \nu_2 \lambda_3 \Pi \\ &- x_{p_1}^4 x_{p_2}^4 \eta_2 \lambda_1 \lambda_2 \nu_3 \rho \nu_1 \delta \varepsilon \eta_2 \lambda_1 \eta_1 \eta_2 \lambda_1 \eta_1 \alpha \nu_2 \lambda_3 \eta_3 \lambda_2 \nu_3 \rho \nu_1 \nu_2 \lambda_3 \Pi \\ &- x_{p_1}^4 x_{p_2}^4 \eta_2 \lambda_1 \lambda_2 \nu_3 \rho \nu_1 \nu_2 \lambda_3 \eta_3 \lambda_2 \nu_3 \rho \nu_1 \delta \varepsilon \eta_2 \lambda_1 \eta_1 \eta_2 \lambda_1 \eta_1 \alpha \nu_2 \lambda_3 \Pi \\ &- x_{p_1}^6 x_{p_2}^6 \eta_2 \lambda_1 \lambda_2 \nu_3 \rho \nu_1 \nu_2 \lambda_3 \Pi \eta_2 \lambda_1 \lambda_2 \nu_3 \rho \nu_1 \delta \varepsilon \eta_2 \lambda_1 \eta_1 \eta_2 \lambda_1 \eta_1 \alpha \nu_2 \lambda_3 \Pi \end{aligned}$$

This proves the Lemma. \square

For the statement of the next lemma, we introduce two pieces of notation. Given a non-zero element u of a complete path algebra $R\langle\langle Q \rangle\rangle$, we denote by $\text{short}(u)$ the largest integer n with the property that $u \in \mathfrak{m}^n$ but $u \notin \mathfrak{m}^{n+1}$. Also, $\ell(c)$ will denote the length of any given path c on Q .

Lemma 6.5. *Let τ be the ideal triangulation defined prior to the statement of Proposition 6.3. Suppose $L \in R\langle\langle Q(\tau) \rangle\rangle$ is a non-zero potential satisfying (6.8). Then there exist a potential $L' \in R\langle\langle Q(\tau) \rangle\rangle$ and a right-equivalence $\varphi : (Q(\tau), W(\tau, \mathbf{x}) + L) \rightarrow (Q(\tau), W(\tau, \mathbf{x}) + L')$, such that:*

- L' satisfies (6.8);
- $\text{short}(L') \geq \text{short}(L)$;
- the number of cycles of length $\text{short}(L)$ that appear in L' is strictly smaller than the number of cycles of length $\text{short}(L)$ that appear in L ;
- φ is a unitriangular automorphism of $R\langle\langle Q(\tau) \rangle\rangle$, and $\text{depth}(\varphi) = \text{short}(L) - 3$.

Proof. Let ξ be a cycle of length $\text{short}(L)$ appearing in L , so that $L = x\xi + \tilde{L}$ for some non-zero scalar x and some potential \tilde{L} . By (6.8), one of the paths $\eta_2 \lambda_1 \lambda_2 \nu_3$ and $\eta_2 \eta_3 \lambda_2 \nu_3$ is a factor of ξ . Rotating ξ if

necessary, we can assume, without loss of generality, that

$$\xi = \begin{cases} \eta_2 \lambda_1 \lambda_2 \nu_3 \zeta & \text{if } \eta_2 \lambda_1 \lambda_2 \nu_3 \text{ is a factor of } \xi; \\ \eta_2 \eta_3 \lambda_2 \nu_3 \zeta & \text{otherwise.} \end{cases}$$

Define an R -algebra automorphism φ of $R\langle\langle Q(\tau) \rangle\rangle$ according to the rule

$$\varphi : \begin{cases} \lambda_3 \mapsto \lambda_3 - x\nu_3\zeta\eta_2 & \text{if } \eta_2 \lambda_1 \lambda_2 \nu_3 \text{ is a factor of } \xi; \\ \eta_1 \mapsto \eta_1 - x\lambda_2\nu_3\zeta & \text{otherwise.} \end{cases}$$

In either case, φ is clearly unitriangular and its depth is equal to $\text{short}(L) - 3$. Set

$$\theta = \begin{cases} \alpha\nu_2\nu_3\zeta\eta_2\eta_3\lambda_2\nu_3\rho\nu_1\beta\gamma\eta_2\lambda_1\eta_1 & \text{if } \eta_2 \lambda_1 \lambda_2 \nu_3 \text{ is a factor of } \xi; \\ \alpha\nu_2\lambda_3\eta_3\lambda_2\nu_3\rho\nu_1\beta\gamma\eta_2\lambda_1\lambda_2\nu_3\zeta & \text{otherwise;} \end{cases}$$

which is in either case a cycle on $Q(\tau)$. Then

$$\varphi(W(\tau, \mathbf{x}) + L) = \varphi(W(\tau, \mathbf{x}) + x\xi + \tilde{L}) \sim_{\text{cyc}} W(\tau, \mathbf{x}) - x\xi + \varphi(x\xi) - x_{p_1}x\theta + \varphi(\tilde{L}).$$

Since $\ell(\zeta) = \ell(\xi) - 4$, the cycle θ certainly has length greater than $\ell(\xi) = \text{short}(L)$. By Lemma 2.3, we have $\varphi(x\xi) - x\xi \in \mathfrak{m}^{2\text{short}(L)-3}$, and since $2\text{short } L - 3 > \text{short } L$, this implies $\text{short}(\varphi(x\xi) - x\xi) > \text{short}(L)$. Again by Lemma 2.3, we also have $\varphi(\tilde{L}) - \tilde{L} \in \mathfrak{m}^{\text{short}(L) + \text{short}(\tilde{L}) - 3}$, and since $\text{short}(L) + \text{short}(\tilde{L}) - 3 > \text{short}(L)$, this means that every cycle of length $\text{short}(L)$ appearing in $\varphi(\tilde{L})$ has to appear in \tilde{L} as well. Therefore, every cycle of length $\text{short}(L)$ appearing in $-x_{p_1}x\theta + \varphi(x\xi) - x\xi + \varphi(\tilde{L})$ has to appear in L as well. Setting $L' = -x_{p_1}x\theta + \varphi(x\xi) - x\xi + \varphi(\tilde{L})$ and noting that the cycle ξ does not appear in L' , we see that $\text{short}(L') \geq \text{short}(L)$ and that the number of cycles of length $\text{short}(L)$ that appear in L' is strictly smaller than the number of cycles of length $\text{short}(L)$ that appear in L .

Let us show that L' satisfies (6.8).

First of all, it is clear that at least one of the paths $\eta_2 \lambda_1 \lambda_2 \nu_3$ and $\eta_2 \eta_3 \lambda_2 \nu_3$ appears as a factor of θ .

Now, if c is a cycle on $Q(\tau)$ satisfying (6.8), that is, if either of the paths $\eta_2 \lambda_1 \lambda_2 \nu_3$ and $\eta_2 \eta_3 \lambda_2 \nu_3$ appears as a factor of c , then every cycle appearing in $\varphi(c)$ has one of the two referred paths as a factor as well. This is a consequence of the following obvious facts:

- φ acts as the identity on every arrow of $Q(\tau)$ different from λ_3 and η_1 ;
- none of the arrows λ_3 and η_1 is involved in any of the two paths $\eta_2 \lambda_1 \lambda_2 \nu_3$ and $\eta_2 \eta_3 \lambda_2 \nu_3$;
- φ is a ring homomorphism, and hence, $\varphi(c)$ can be obtained from c by applying φ to it *arrow by arrow*, that is, by replacing a with $\varphi(a)$ for every arrow a appearing in c .

This implies that $\varphi(x\xi) - x\xi + \varphi(\tilde{L})$ satisfies (6.8), since ξ and \tilde{L} satisfy (6.8).

Lemma 6.5 is proved. \square

Proof of Proposition 6.3. By Lemma 2.4, it suffices to show the existence of a sequence $(L_n)_{n>0}$ of potentials on $Q(\tau)$ and a sequence $(\varphi_n)_{n\geq 0}$ of R -algebra automorphisms of $R\langle\langle Q(\tau) \rangle\rangle$ such that:

$$(6.9) \quad \lim_{n \rightarrow \infty} L_n = 0;$$

$$(6.10) \quad \varphi_n \text{ is unitriangular for all } n > 0, \text{ and } \lim_{n \rightarrow \infty} \text{depth}(\varphi_n) = \infty;$$

$$(6.11) \quad \varphi_0 \text{ is a right-equivalence } (Q(\tau), S(\tau, \mathbf{x})) \rightarrow (Q(\tau), W(\tau, \mathbf{x}) + L_1);$$

$$(6.12) \quad \varphi_n \text{ is a right-equivalence } (Q(\tau), W(\tau, \mathbf{x}) + L_n) \rightarrow (Q(\tau), W(\tau, \mathbf{x}) + L_{n+1}) \text{ for all } n > 0.$$

Indeed, Lemma 2.4 guarantees that, if such sequences exist, then the limit $\lim_{n \rightarrow \infty} \varphi_n \varphi_{n-1} \dots \varphi_2 \varphi_1 \varphi_0$ is a right-equivalence $(Q(\tau), S(\tau, \mathbf{x})) \rightarrow (Q(\tau), W(\tau, \mathbf{x}))$.

By Lemma 6.4, there exists a right-equivalence $\varphi_0 : (Q(\tau), S(\tau, \mathbf{x})) \rightarrow (Q(\tau), W(\tau, \mathbf{x}) + L_1)$, where $L_1 \in R\langle\langle Q(\tau) \rangle\rangle$ is a potential satisfying (6.8).

Now, given a potential $L_n \in R\langle\langle Q(\tau) \rangle\rangle$ satisfying (6.8), Lemma 6.5 guarantees the existence of a potential $L_{n+1} \in R\langle\langle Q(\tau) \rangle\rangle$ and a right-equivalence $\varphi_n : (Q(\tau), W(\tau, \mathbf{x}) + L_n) \rightarrow (Q(\tau), W(\tau, \mathbf{x}) + L_{n+1})$, such that:

- L_{n+1} satisfies (6.8);
- $\text{short}(L_{n+1}) \geq \text{short}(L_n)$;
- the number of cycles of length $\text{short}(L_n)$ that appear in L_{n+1} is strictly smaller than the number of cycles of length $\text{short}(L_n)$ that appear in L_n ;

- φ_n is unitriangular and $\text{depth}(\varphi_n) = \text{short}(L_n) - 3$.

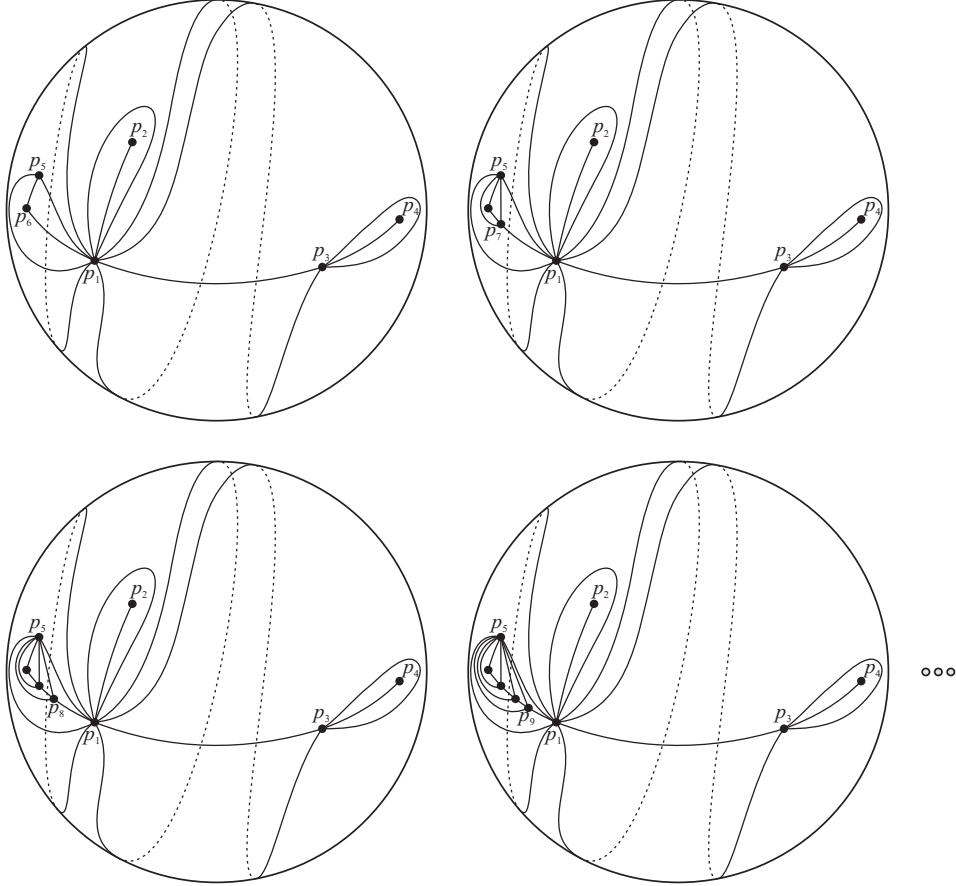
Consider the sequences $(L_n)_{n>0}$ and $(\varphi_n)_{n\geq 0}$ thus constructed. The last three of the four properties just listed, together with the fact that for any given $\ell > 0$ the quiver $Q(\tau)$ admits only finitely many cycles of length ℓ , imply $\lim_{n\rightarrow\infty} L_n = 0$ and $\lim_{n\rightarrow\infty} \text{depth}(\varphi_n) = \infty$. Therefore, the sequences $(L_n)_{n>0}$ and $(\varphi_n)_{n\geq 0}$ satisfy all desired properties (6.9), (6.10), (6.11) and (6.12).

Proposition 6.3 is proved. \square

We have thus proved that every positive-genus surface with empty boundary and at least two punctures admits an ideal triangulation τ with a self-folded triangle for which the QPs $(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\tau), W(\tau, \mathbf{x}))$ are right-equivalent. This completes the proof of the Popping Theorem 6.1 in the case of empty-boundary surfaces of positive genus.

6.3. Proof for genus-zero surfaces with empty boundary. In this subsection we prove Proposition 6.2 in the case when (Σ, \mathbb{M}) is a sphere with at least 6 punctures. To do so it suffices to exhibit an ideal triangulation τ of (Σ, \mathbb{M}) having a self-folded triangle for which the corresponding QPs $(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\tau), W(\tau, \mathbf{x}))$ are right-equivalent (this is because Σ is assumed to have empty boundary –cf. the considerations made right at the beginning of Subsection 6.2). We construct such triangulation by induction on the number $|\mathbb{M}| \geq 6$ of punctures as sketched in Figure 17.

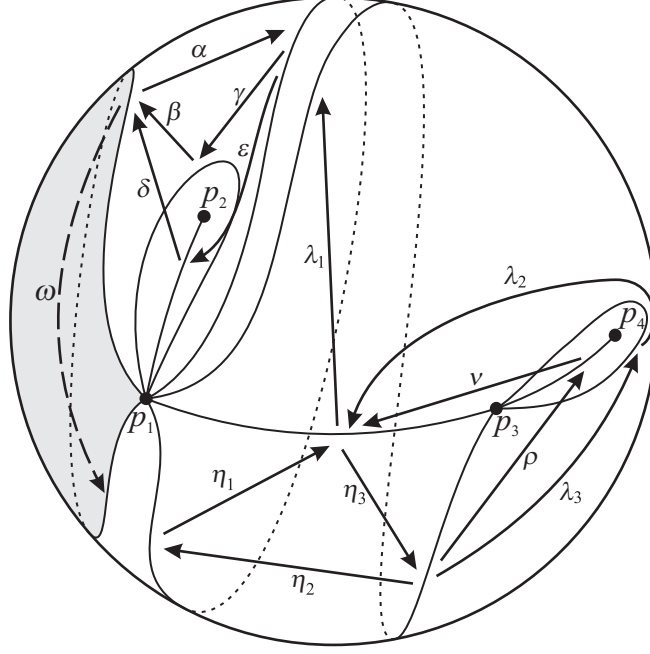
FIGURE 17. Sketch of the induction used to construct τ . The induction is on the number $|\mathbb{M}| \geq 6$ of punctures of (Σ, \mathbb{M})



The quiver of τ is depicted in Figure 18. Using the notation of that same Figure, the potentials $S(\tau, \mathbf{x})$ and $W(\tau, \mathbf{x})$ are

$$\begin{aligned} S(\tau, \mathbf{x}) &= -x_{p_2}^{-1} \alpha \delta \varepsilon + \eta_1 \eta_2 \eta_3 + \lambda_1 \lambda_2 \lambda_3 + \alpha \beta \gamma + x_{p_1} \alpha \omega \delta \varepsilon \eta_2 \lambda_1 \eta_1 + x_{p_3} \eta_3 \nu \rho - x_{p_4}^{-1} \lambda_1 \nu \rho + S'(\tau), \\ W(\tau, \mathbf{x}) &= \alpha \delta \varepsilon + \eta_1 \eta_2 \eta_3 + \lambda_1 \lambda_2 \lambda_3 + x_{p_2}^{-1} \alpha \beta \gamma + x_{p_1} \alpha \omega \beta \gamma \eta_2 \lambda_1 \eta_1 + x_{p_3} \eta_3 \nu \rho - x_{p_4}^{-1} \lambda_1 \nu \rho + S'(\tau), \end{aligned}$$

FIGURE 18. The quiver $Q(\tau)$. The shading indicates that several punctures, arcs and arrows contained in the shaded area have been omitted.



where $S'(\tau)$ is a potential not involving any of the arrows $\alpha, \beta, \gamma, \delta, \varepsilon, \eta_1, \eta_2, \eta_3, \lambda_1, \lambda_2, \lambda_3$.

Proposition 6.6. *For the ideal triangulation τ just defined, the QPs $(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\tau), W(\tau, \mathbf{x}))$ are right-equivalent.*

The proof of Proposition 6.6 will follow from the next two lemmas.

Lemma 6.7. *For the ideal triangulation τ above, the QP $(Q(\tau), S(\tau, \mathbf{x}))$ is right-equivalent to a QP of the form $(Q(\tau), W(\tau, \mathbf{x}) + L_1)$, where $L_1 \in R\langle\langle Q(\tau) \rangle\rangle$ is a potential satisfying the following condition:*

(6.13) *every cycle appearing in it has at least one of the paths $\eta_2 \lambda_1 \nu \rho$ and $\eta_2 \eta_3 \nu \rho$ as a factor.*

Proof. Define R -algebra automorphisms $\psi_{\beta, \delta}, \psi_{\alpha}, \psi_{\eta_3} : R\langle\langle Q(\tau) \rangle\rangle \rightarrow R\langle\langle Q(\tau) \rangle\rangle$ according to the rules

$$\begin{aligned} \psi_{\beta, \delta} &: \beta \mapsto x_{p_2}^{-1} \beta, \quad \delta \mapsto -x_{p_2} \delta, \\ \psi_{\alpha} &: \alpha \mapsto \alpha + x_{p_1} x_{p_2} \eta_2 \lambda_1 \eta_1 \alpha \omega \\ \psi_{\eta_3} &: \eta_3 \mapsto \eta_3 + x_{p_1}^2 x_{p_2}^2 \lambda_1 \eta_1 \alpha \omega^2 \delta \varepsilon \eta_2 \lambda_1. \end{aligned}$$

A straightforward calculation shows that

$$\psi_{\eta_3} \psi_{\alpha} \psi_{\beta, \delta}(S(\tau, \mathbf{x})) \sim_{\text{cyc}} W(\tau, \mathbf{x}) + x_{p_1}^2 x_{p_2}^2 x_{p_3} \eta_2 \lambda_1 \nu \rho \lambda_1 \eta_1 \alpha \omega^2 \delta \varepsilon$$

This proves the Lemma. \square

Lemma 6.8. *Let τ be the ideal triangulation defined prior to the statement of Proposition 6.6. Suppose $L \in R\langle\langle Q(\tau) \rangle\rangle$ is a non-zero potential satisfying (6.13). Then there exist a potential $L' \in R\langle\langle Q(\tau) \rangle\rangle$ and a right-equivalence $\varphi : (Q(\tau), W(\tau, \mathbf{x}) + L) \rightarrow (Q(\tau), W(\tau, \mathbf{x}) + L')$, such that:*

- L' satisfies (6.13);
- $\text{short}(L') \geq \text{short}(L)$;
- the number of cycles of length $\text{short}(L)$ that appear in L' is strictly smaller than the number of cycles of length $\text{short}(L)$ that appear in L ;
- φ is a unitriangular automorphism of $R\langle\langle Q(\tau) \rangle\rangle$, and $\text{depth}(\varphi) = \text{short}(L) - 3$.

Proof. Let ξ be a cycle of length $\text{short}(L)$ appearing in L , so that $L = x\xi + \tilde{L}$ for some non-zero scalar x and some potential \tilde{L} . By (6.13), one of the paths $\eta_2\lambda_1\nu\rho$ and $\eta_2\eta_3\nu\rho$ is a factor of ξ . Suppose $\eta_2\eta_3\nu\rho$ is a factor of ξ . Rotating ξ if necessary, we can assume, without loss of generality, that

$$\xi = \eta_2\eta_3\nu\rho\zeta.$$

Define an R -algebra automorphism φ of $R\langle\langle Q(\tau) \rangle\rangle$ according to the rule

$$\varphi : \eta_1 \mapsto \eta_1 - x\nu\rho\zeta.$$

This automorphism φ is clearly unitriangular and its depth is equal to $\text{short}(L) - 3$. Set

$$\theta = \alpha\omega\beta\gamma\eta_2\lambda_1\nu\rho\zeta$$

which is obviously a cycle on $Q(\tau)$. Then

$$\varphi(W(\tau, \mathbf{x}) + L) = \varphi(W(\tau, \mathbf{x}) + x\xi + \tilde{L}) \sim_{\text{cyc}} W(\tau, \mathbf{x}) - x\xi + \varphi(x\xi) + xx_{p_1}\theta + \varphi(\tilde{L})$$

Since $\ell(\zeta) = \ell(\xi) - 4$, the cycle θ certainly has length greater than $\ell(\xi) = \text{short}(L)$. By Lemma 2.3, we have $\varphi(x\xi) - x\xi \in \mathfrak{m}^{2\text{short}(L)-3}$, and since $2\text{short}(L) - 3 > \text{short}(L)$, this implies $\text{short}(\varphi(x\xi) - x\xi) > \text{short}(L)$. Again by Lemma 2.3, we also have $\varphi(\tilde{L}) - \tilde{L} \in \mathfrak{m}^{\text{short}(L) + \text{short}(\tilde{L}) - 3}$, and since $\text{short}(L) + \text{short}(\tilde{L}) - 3 > \text{short}(L)$, this means that every cycle of length $\text{short}(L)$ appearing in $\varphi(\tilde{L})$ has to appear in \tilde{L} as well. Therefore, every cycle of length $\text{short}(L)$ appearing in $-x_{p_1}\theta + \varphi(x\xi) - x\xi + \varphi(\tilde{L})$ has to appear in L as well. Setting $L' = -x_{p_1}\theta + \varphi(x\xi) - x\xi + \varphi(\tilde{L})$ and noting that the cycle ξ does not appear in L' , we see that $\text{short}(L') \geq \text{short}(L)$ and that the number of cycles of length $\text{short}(L)$ that appear in L' is strictly smaller than the number of cycles of length $\text{short}(L)$ that appear in L .

Let us show that L' satisfies (6.13).

First of all, it is clear that at least one of the paths $\eta_2\lambda_1\nu\rho$ and $\eta_2\eta_3\nu\rho$ appears as a factor of θ .

Now, if c is a cycle on $Q(\tau)$ satisfying (6.13), that is, if either of the paths $\eta_2\lambda_1\nu\rho$ and $\eta_2\eta_3\nu\rho$ appears as a factor of c , then every cycle appearing in $\varphi(c)$ has one of the two referred paths as a factor as well. This is a consequence of the following obvious facts:

- φ acts as the identity on every arrow of $Q(\tau)$ different from η_1 ;
- the arrow η_1 is not involved in any of the two paths $\eta_2\lambda_1\nu\rho$ and $\eta_2\eta_3\nu\rho$;
- φ is a ring homomorphism, and hence, $\varphi(c)$ can be obtained from c by applying φ to it *arrow by arrow*, that is, by replacing a with $\varphi(a)$ for every arrow a appearing in c .

This implies that $\varphi(x\xi) - x\xi + \varphi(\tilde{L})$ satisfies (6.13), since ξ and \tilde{L} satisfy (6.13).

Thus far, we have shown that if the path $\eta_2\eta_3\nu\rho$ is a factor of ξ , then there exists a potential L' and a right-equivalence $\varphi : (Q(\tau), W(\tau, \mathbf{x}) + L) \rightarrow (Q(\tau), W(\tau, \mathbf{x}) + L')$ satisfying the properties stated in Lemma 6.8.

Now, suppose that $\eta_2\lambda_1\nu\rho$ is a factor of ξ and $\eta_2\eta_3\nu\rho$ is not. From what has been shown so far in the ongoing proof, we deduce that in order to prove the existence of a desired L' and a desired φ in this case, it suffices to show the existence of a potential $L'' \in R\langle\langle Q(\tau) \rangle\rangle$ and a right-equivalence $\psi : (Q(\tau), W(\tau, \mathbf{x}) + L) \rightarrow (Q(\tau), W(\tau, \mathbf{x}) + L'')$, with the following properties:

- L'' satisfies (6.13);
- $\text{short}(L'') \geq \text{short}(L)$;
- the number of cycles of length $\text{short}(L)$ that appear in L'' does not exceed the number of cycles of length $\text{short}(L)$ that appear in L ;
- the number of cycles of length $\text{short}(L)$ that appear in L'' and have $\eta_2\eta_3\nu\rho$ as a factor is strictly bigger than the number of cycles of length $\text{short}(L)$ that appear in L and have $\eta_2\eta_3\nu\rho$ as a factor;
- ψ is a unitriangular automorphism of $R\langle\langle Q(\tau) \rangle\rangle$, and $\text{depth}(\psi) = \text{short}(L) - 3$.

The rest of the proof is devoted to show the existence of such an L'' and such a ψ .

Rotating ξ if necessary, we can assume, without loss of generality, that

$$\xi = \eta_2\lambda_1\nu\rho\zeta$$

Define an R -algebra automorphism ψ of $R\langle\langle Q(\tau) \rangle\rangle$ according to the rule

$$\psi : \rho \mapsto \rho + xx_{p_4}\rho\zeta\eta_2.$$

This automorphism ψ is clearly unitriangular and its depth is equal to $\text{short}(L) - 3$. Set

$$\theta = \eta_2 \eta_3 \nu \rho \zeta,$$

which is a cycle on $Q(\tau)$. Then

$$\psi(W(\tau, \mathbf{x}) + L) = \psi(W(\tau, \mathbf{x}) + x\xi + \tilde{L}) \sim_{\text{cyc}} W(\tau, \mathbf{x}) - x\xi + \psi(x\xi) + xx_{p_3}x_{p_4}\theta + \psi(\tilde{L}).$$

Since $\ell(\zeta) = \ell(\xi) - 4$, the cycle θ certainly has length equal to $\ell(\xi) = \text{short}(L)$. By Lemma 2.3, we have $\psi(x\xi) - x\xi \in \mathfrak{m}^{2\text{short}(L)-3}$, and since $2\text{short}(L) - 3 > \text{short}(L)$, this implies $\text{short}(\psi(x\xi) - x\xi) > \text{short}(L)$. Again by Lemma 2.3, we also have $\varphi(\tilde{L}) - \tilde{L} \in \mathfrak{m}^{\text{short}(L)+\text{short}(\tilde{L})-3}$, and since $\text{short}(L) + \text{short}(\tilde{L}) - 3 > \text{short}(L)$, this means that every cycle of length $\text{short}(L)$ appearing in $\psi(\tilde{L})$ has to appear in \tilde{L} as well. Setting $L'' = xx_{p_3}x_{p_4}\theta + \psi(x\xi) - x\xi + \psi(\tilde{L})$ and noting that the cycle ξ does not appear in L'' , we see that $\text{short}(L') \geq \text{short}(L)$ and that the number of cycles of length $\text{short}(L)$ that appear in L'' does not exceed the number of cycles of length $\text{short}(L)$ that appear in L .

Let us show that L'' satisfies (6.8).

First of all, it is clear that at least one of the paths $\eta_2\lambda_1\nu\rho$ and $\eta_2\eta_3\nu\rho$ appears as a factor of θ .

Now, if c is a cycle on $Q(\tau)$ satisfying (6.13), that is, if either of the paths $\eta_2\lambda_1\nu\rho$ and $\eta_2\eta_3\nu\rho$ appears as a factor of c , then every cycle appearing in $\psi(c)$ has one of the two referred paths as a factor as well. This is a consequence of the following obvious facts:

- ψ acts as the identity on every arrow of $Q(\tau)$ different from ρ ;
- $\psi(\eta_2\lambda_1\nu\rho) = \eta_2\lambda_1\nu\rho + xx_{p_4}\eta_2\lambda_1\nu\rho\zeta\eta_2$ and $\psi(\eta_2\eta_3\nu\rho) = \eta_2\eta_3\nu\rho + xx_{p_4}\eta_2\eta_3\nu\rho\zeta\eta_2$;
- φ is a ring homomorphism, and hence, $\varphi(c)$ can be obtained from c by applying φ to it *arrow by arrow*, that is, by replacing a with $\varphi(a)$ for every arrow a appearing in c .

This implies that $\varphi(x\xi) - x\xi + \varphi(\tilde{L})$ satisfies (6.8), since ξ and \tilde{L} satisfy (6.13).

Finally, since $\eta_2\eta_3\nu\rho$ is obviously a factor of $\theta = \eta_2\eta_3\nu\rho\zeta$, the number of cycles of length $\text{short}(L)$ that appear in L'' and have $\eta_2\eta_3\nu\rho$ as a factor is strictly bigger than the number of cycles of length $\text{short}(L)$ that appear in L and have $\eta_2\eta_3\nu\rho$ as a factor.

Lemma 6.8 is proved. \square

Proof of Proposition 6.6. This follows from a combination of Lemmas 6.7, 6.8 and 2.4, similar to the combination of Lemmas 6.4, 6.5 and 2.4 that yields Proposition 6.3. \square

The proof of the Popping Theorem is now complete in the case of all empty-boundary surfaces different from a sphere with less than 6 punctures.

6.4. Proof for surfaces with non-empty boundary. For surfaces with non-empty boundary the Popping Theorem can be proved in at least two different ways. In this subsection we give a proof via *restriction*, using the fact that by now we already know that the Popping theorem holds for all empty-boundary surfaces satisfying (6.1). A direct proof can be given, without going through empty-boundary considerations, by “pulling undesired terms to the boundary”.

Let us recall the definition of restriction and a couple of results from [15] and [16].

Definition 6.9 ([10, Definition 8.8]). Let (Q, S) be any QP (not necessarily arising from a surface) and I be a subset of the vertex set Q_0 . The *restriction* of (Q, S) to I is the QP $(Q|_I, S|_I)$, where

- $Q|_I$ is the quiver obtained from Q by deleting the arrows incident to elements from $Q_0 \setminus I$, the vertex set of $Q|_I$ is Q_0 ;
- $S|_I$ is the image of S under the R -algebra homomorphism $\psi_I : R\langle\langle A \rangle\rangle \rightarrow R\langle\langle A|_I \rangle\rangle$ defined by the rule $\psi_I(a) = a$ for every arrow a of Q whose head and tail simultaneously belong to I , and $\psi_I(b) = 0$ for every arrow b incident to at least one element of $Q_0 \setminus I$.

Lemma 6.10 ([16, Proof of Lemma 2.30]). *Suppose $\varphi : (Q, S) \rightarrow (Q', S')$ is a right-equivalence (with the QPs (Q, S) and (Q', S') not necessarily arising from a surface). If I is a subset of Q_0 , then the R -algebra homomorphism $\varphi|_I : R\langle\langle Q|_I \rangle\rangle \rightarrow R\langle\langle Q'|_I \rangle\rangle$ defined by the rule $u \mapsto \varphi(u)|_I$ is a right-equivalence between $(Q|_I, S|_I)$ and $(Q'|_I, S'|_I)$.*

The following two lemmas will be proved simultaneously in the proof of Proposition 6.13.

Lemma 6.11 ([15, Lemma 29]). *For every QP of the form $(Q(\tau), S(\tau, \mathbf{x}))$ with τ an ideal triangulation there exists an ideal triangulation $\tilde{\tau}$ of a surface $(\tilde{\Sigma}, \tilde{\mathbb{M}})$ with empty boundary, with the following properties:*

- $\tilde{\tau}$ contains all the arcs of τ ;
- $(Q(\tau), S(\tau, \mathbf{x}))$ can be obtained from the restriction of $(Q(\tilde{\tau}), S(\tilde{\tau}, \tilde{\mathbf{x}}))$ to τ by deleting the elements of $\tilde{\tau} \setminus \tau$, where $\tilde{\mathbf{x}} = (x_q)_{q \in \tilde{\mathbb{M}}}$ is any extension of the choice $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ to $\tilde{\mathbb{M}}$.

Lemma 6.12. *For every QP of the form $(Q(\tau), W(\tau, \mathbf{x}))$ with τ an ideal triangulation and $W(\tau, \mathbf{x})$ the popped potential associated to a quadruple (τ, \mathbf{x}, i, j) , there exists an ideal triangulation $\tilde{\tau}$ of a surface $(\tilde{\Sigma}, \tilde{\mathbb{M}})$ with empty boundary with the following properties:*

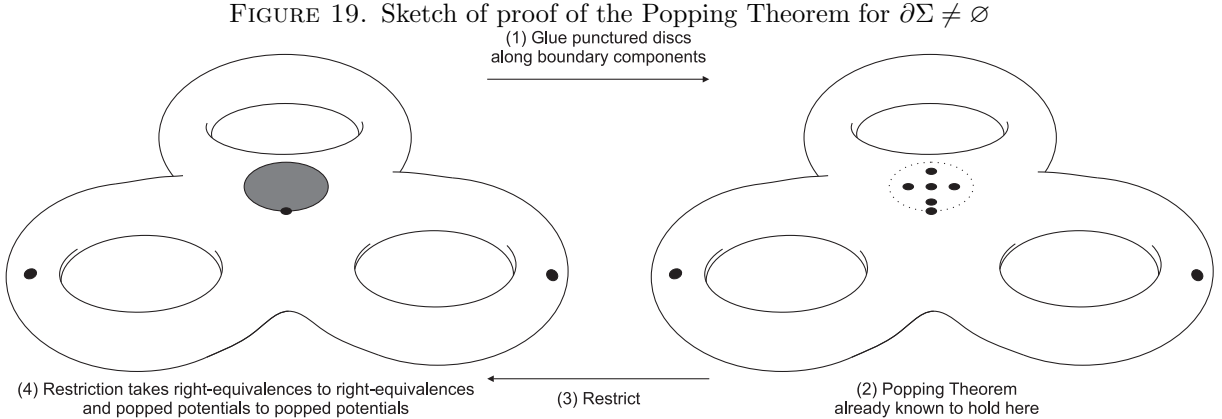
- $\tilde{\tau}$ contains all the arcs of τ ;
- $(Q(\tau), W(\tau, \mathbf{x}))$ can be obtained from the restriction of $(Q(\tilde{\tau}), W(\tilde{\tau}, \tilde{\mathbf{x}}))$ to τ by deleting the elements of $\tilde{\tau} \setminus \tau$, where $\tilde{\mathbf{x}} = (x_q)_{q \in \tilde{\mathbb{M}}}$ is any extension of the choice $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ to $\tilde{\mathbb{M}}$ and $W(\tilde{\tau}, \tilde{\mathbf{x}})$ is the popped potential associated to the quadruple $(\tilde{\tau}, \tilde{\mathbf{x}}, i, j)$.

Proposition 6.13. *Proposition 6.2 holds for surfaces with non-empty boundary.*

Proof. Let i, j be as in the statement of Proposition 6.2, and let τ be any ideal triangulation of (Σ, \mathbb{M}) containing i and j . For each boundary component b of Σ , let m_b be the number of marked points lying on b . Since each b is homeomorphic to a circle, we can glue Σ and a triangulated 5-punctured m_b -gon along b . Making such gluing along every boundary component of Σ will result in a surface $(\tilde{\Sigma}, \tilde{\mathbb{M}})$ with empty boundary and more than 5 punctures, and an ideal triangulation $\tilde{\tau}$ containing τ . In particular, $i, j \in \tilde{\tau}$. It is easy to check that the ideal triangulation $\tilde{\tau}$ satisfies the properties stated in Lemmas 6.11 and 6.12.

Since Theorem 6.1 is already known to hold in the empty boundary case (cf. Subsections 6.2 and 6.3), the pop of (i, j) in $\tilde{\tau}$ induces right-equivalence, that is, there exists a right-equivalence between $(Q(\tilde{\tau}), S(\tilde{\tau}, \tilde{x}))$ and $(Q(\tilde{\tau}), W(\tilde{\tau}, \tilde{x}))$. Since $\tilde{\tau}$ simultaneously proves Lemmas 6.11 and 6.12, a straightforward combination of this fact with Lemma 6.10 finishes the proof of Proposition 6.13. \square

Example 6.14. In Figure 19 we have sketched the proof of Proposition 6.13. \blacktriangle



7. LEAVING THE POSITIVE STRATUM

Let (Σ, \mathbb{M}) be any surface different from a 5-punctured sphere. Thus, (Σ, \mathbb{M}) is allowed to be any of the surfaces listed in (6.2), (6.3) and (6.4).

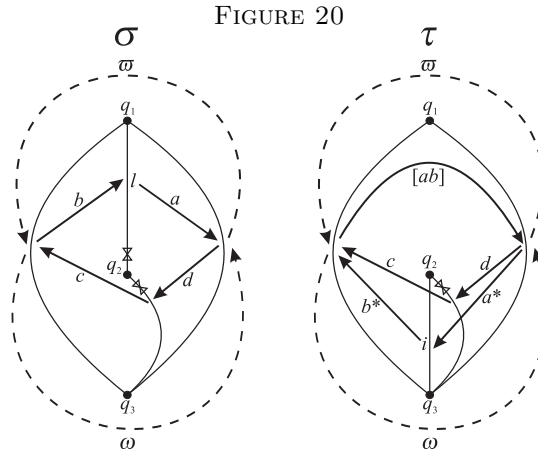
Ideal triangulations are precisely the tagged triangulations that have non-negative signature, and these are precisely the tagged triangulations that have positive weak signature. Thus, in any tagged triangulation with non-negative signature we have a well-defined notion of *folded side*. It is the flips of folded sides the ones that have proven to be extremely difficult to deal with, at least in regard to QP-mutations.

Theorem 7.1. *Let τ be any ideal triangulation of (Σ, \mathbb{M}) , and let i be any arc belonging to τ . Then $\mu_i(Q(\tau), S(\tau, \mathbf{x}))$ is right-equivalent to $(Q(\sigma), S(\sigma, \mathbf{x}))$.*

Proof. By Theorem 3.6 we can assume that i is a folded side of τ . Note that the loop j that cuts out a once-punctured monogon and encloses i can be identified with $t_{\epsilon_\sigma}(i)$. Let l be the unique tagged arc such that $\sigma = (\tau \setminus \{i\}) \cup \{l\}$. Since QP-mutations are involutive up to right-equivalence, to prove Theorem 7.1 it suffices to show that $\mu_l(Q(\sigma), S(\sigma, \mathbf{x}))$ and $(Q(\tau), S(\tau, \mathbf{x}))$ are right-equivalent. A direct computation shows that $\mu_l(Q(\sigma), S(\sigma, \mathbf{x}))$ is right-equivalent to $(Q(\tau), W(\tau, \mathbf{x}))$, where $W(\tau, \mathbf{x})$ is the popped potential of the quadruple (τ, \mathbf{x}, i, j) (this computation is made in Examples 7.2, 7.3, 7.4, 7.5 and 7.6 below). By the Popping Theorem 6.1, $(Q(\tau), W(\tau, \mathbf{x}))$ and $(Q(\tau), S(\tau, \mathbf{x}))$ are right-equivalent. Theorem 7.1 is proved. \square

In the examples to follow we will keep the notations used in the proof of Theorem 7.1.

Example 7.2. Suppose that the configurations that σ and τ respectively present around l and i are the ones shown in Figure 20. Assuming that $q_1 \neq q_3$ and that the path ϖ has length greater than 1, we have



$$S(\sigma, \mathbf{x}) = x_{q_1} ab\varpi + x_{q_2}^{-1} abcd + x_{q_3} cd\omega + S'(\sigma) \text{ and}$$

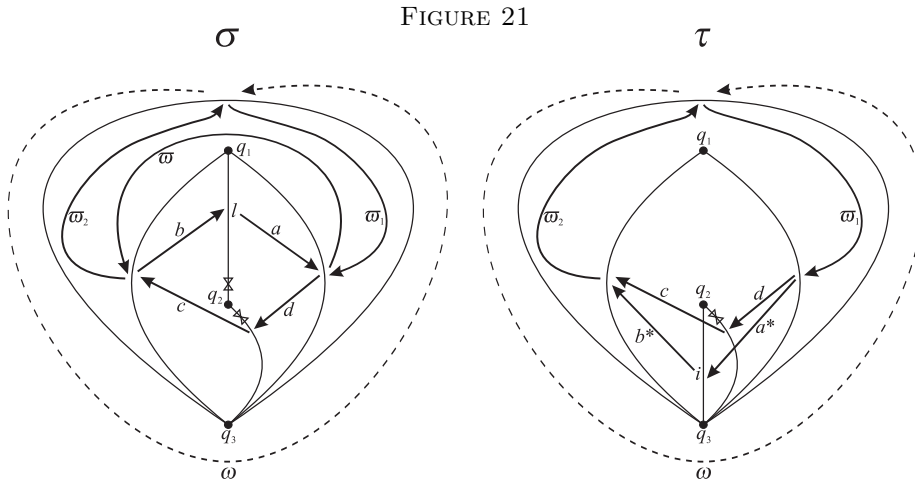
$$W(\tau, \mathbf{x}) = a^*[ab]b^* + x_{q_1} [ab]\varpi + x_{q_2}^{-1} [ab]cd + x_{q_3} cd\omega + S'(\sigma).$$

Furthermore, $\mu_l(Q(\sigma)) = \tilde{\mu}_l(Q(\sigma)) = Q(\tau)$ (with the vertex $i \in Q(\tau)$ replacing the vertex $l \in Q(\sigma)$) and

$$\mu_l(S(\sigma, \mathbf{x})) = \tilde{\mu}_l(S(\sigma, \mathbf{x})) = x_{q_1} [ab]\varpi + x_{q_2}^{-1} [ab]cd + x_{q_3} cd\omega + S'(\sigma) + a^*[ab]b^*.$$

The QPs $\mu_l(Q(\sigma), S(\sigma, \mathbf{x}))$ and $(Q(\tau), W(\tau, \mathbf{x}))$ are then obviously right-equivalent. \blacktriangle

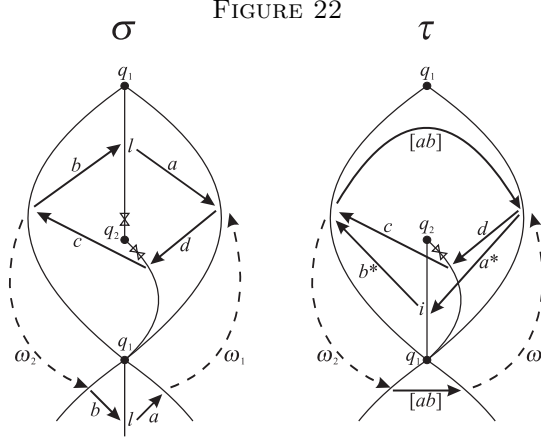
Example 7.3. Suppose that the configurations that σ and τ respectively present around l and i are the ones shown in Figure 21. Assuming that $q_1 \neq q_3$ (and that the path ϖ has length 1), we have



$$\begin{aligned}
S(\sigma, \mathbf{x}) &= \varpi_1 \varpi_2 \varpi + x_{q_1} ab \varpi + x_{q_2}^{-1} abcd + x_{q_3} cd \varpi_1 \varpi_2 + S'(\sigma), \\
W(\tau, \mathbf{x}) &= -x_1^{-1} \varpi_1 \varpi_2 b^* a^* - x_1^{-1} x_2^{-1} \varpi_1 \varpi_2 cd + x_3 cd \varpi_1 \varpi_2 + S'(\sigma), \text{ and} \\
\mu_l(S(\sigma, \mathbf{x})) &= -x_1^{-1} \varpi_1 \varpi_2 b^* a^* - x_1^{-1} x_2^{-1} \varpi_1 \varpi_2 cd + x_3 cd \varpi_1 \varpi_2 + S'(\sigma).
\end{aligned}$$

The QPs $\mu_l(Q(\sigma), S(\sigma, \mathbf{x}))$ and $(Q(\tau), W(\tau, \mathbf{x}))$ are then obviously right-equivalent. \blacktriangle

Example 7.4. Suppose that the configurations that σ and τ respectively present around l and i are the ones shown in Figure 22. We have



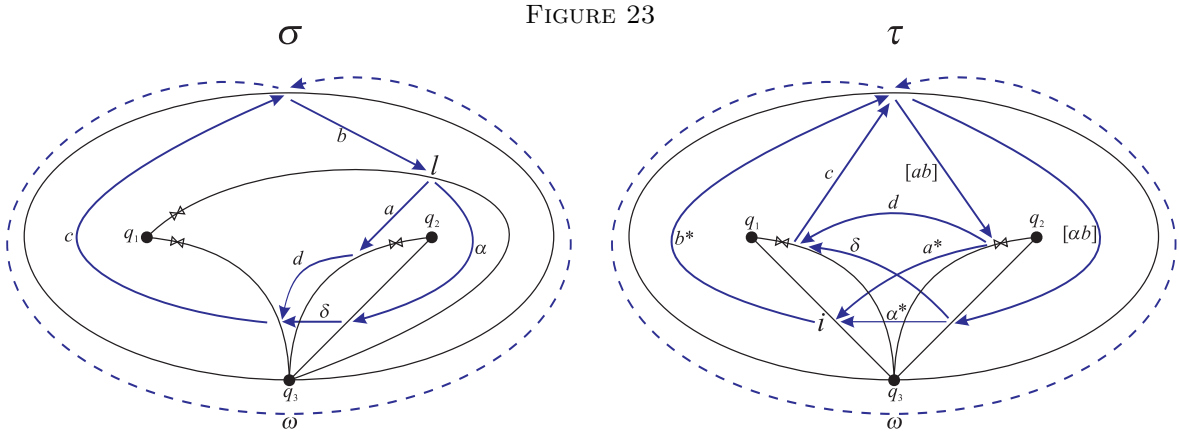
$$\begin{aligned}
S(\sigma, \mathbf{x}) &= x_{q_1} \omega_1 ab \omega_2 cd + x_{q_2}^{-1} abcd + S'(\sigma) \text{ and} \\
W(\tau, \mathbf{x}) &= a^* [ab] b^* + x_{q_1} \omega_1 [ab] \omega_2 cd + x_{q_2}^{-1} d [ab] c + S'(\sigma).
\end{aligned}$$

Furthermore, $\mu_l(Q(\sigma)) = \tilde{\mu}_l(Q(\sigma)) = Q(\tau)$ (with the vertex $i \in Q(\tau)$ replacing the vertex $l \in Q(\sigma)$) and

$$\mu_l(S(\sigma, \mathbf{x})) = x_{q_1} \omega_1 [ab] \omega_2 cd + x_{q_2}^{-1} [ab] cd + S'(\sigma) + a^* [ab] b^*.$$

The QPs $\mu_l(Q(\sigma), S(\sigma, \mathbf{x}))$ and $(Q(\tau), W(\tau, \mathbf{x}))$ are then obviously right-equivalent. \blacktriangle

Example 7.5. Suppose that the configurations that σ and τ respectively present around l and i are the ones shown in Figure 23. We have



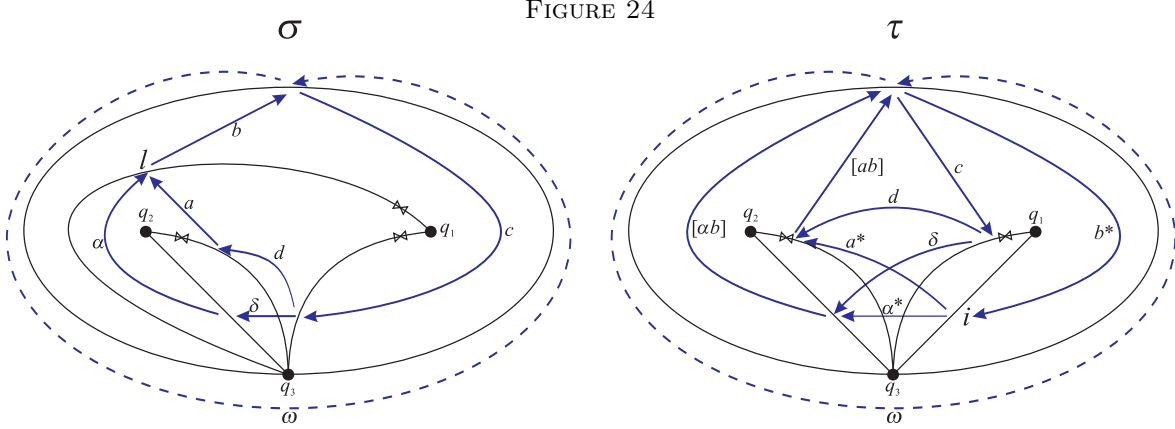
$$\begin{aligned}
S(\sigma, \mathbf{x}) &= x_{q_1}^{-1} abcd - x_{q_1}^{-1} x_{q_2}^{-1} \alpha bcd + x_{q_3} \alpha b \omega c \delta + S'(\sigma) \text{ and} \\
W(\tau, \mathbf{x}) &= a^* [ab] b^* + x_{q_1}^{-1} d [ab] c - x_{q_2} \alpha^* [\alpha b] b^* - x_{q_1}^{-1} x_{q_2}^{-1} \delta [\alpha b] c + x_{q_3} \delta [\alpha b] \omega c + S'(\sigma).
\end{aligned}$$

Furthermore, $\mu_l(Q(\sigma)) = \tilde{\mu}_l(Q(\sigma)) = Q(\tau)$ (with the vertex $i \in Q(\tau)$ replacing the vertex $l \in Q(\sigma)$) and

$$\tilde{\mu}_l(S(\sigma, \mathbf{x})) = \mu_l(S(\sigma, \mathbf{x})) = x_{q_1}^{-1}[ab]cd - x_{q_1}^{-1}x_{q_2}^{-1}[\alpha b]c\delta + x_{q_3}[ab]\omega c\delta + S'(\sigma) + a^*[ab]b^* + \alpha^*[\alpha b]b^*.$$

The R -algebra isomorphism $\varphi : R\langle\langle\mu_l(Q(\sigma))\rangle\rangle \rightarrow R\langle\langle Q(\tau)\rangle\rangle$ acting by $\alpha^* \mapsto -x_{q_2}\alpha^*$ and the identity on the rest of the arrows, is easily seen to be a right-equivalence $\varphi : \mu_l(Q(\sigma), S(\sigma, \mathbf{x})) \rightarrow (Q(\tau), W(\tau, \mathbf{x}))$. \blacktriangle

Example 7.6. Suppose that the configurations that σ and τ respectively present around l and i are the ones shown in Figure 24. We have



$$S(\sigma, \mathbf{x}) = x_{q_1}^{-1}dcba - x_{q_1}^{-1}x_{q_2}^{-1}\delta cb\alpha + x_{q_3}\delta c\omega b\alpha + S'(\sigma) \text{ and}$$

$$W(\tau, \mathbf{x}) = b^*[ba]a^* + x_{q_1}^{-1}c[ba]d - x_{q_2}b^*[ba]\alpha^* - x_{q_1}^{-1}x_{q_2}^{-1}c[ba]\delta + x_{q_3}c\omega[ba]\delta + S'(\sigma).$$

Furthermore, $\mu_l(Q(\sigma)) = \tilde{\mu}_l(Q(\sigma)) = Q(\tau)$ (with the vertex $i \in Q(\tau)$ replacing the vertex $l \in Q(\sigma)$) and

$$\tilde{\mu}_l(S(\sigma, \mathbf{x})) = \mu_l(S(\sigma, \mathbf{x})) = x_{q_1}^{-1}dc[ba] - x_{q_1}^{-1}x_{q_2}^{-1}\delta c[ba] + x_{q_3}\delta c\omega[ba] + S'(\sigma) + b^*[ba]a^* + b^*[ba]\alpha^*.$$

The R -algebra isomorphism $\varphi : R\langle\langle\mu_l(Q(\sigma))\rangle\rangle \rightarrow R\langle\langle Q(\tau)\rangle\rangle$ acting by $\alpha^* \mapsto -x_{q_2}\alpha^*$ and the identity on the rest of the arrows, is easily seen to be a right-equivalence $\varphi : \mu_l(Q(\sigma), S(\sigma, \mathbf{x})) \rightarrow (Q(\tau), W(\tau, \mathbf{x}))$. \blacktriangle

Remark 7.7. Example 7.6 can be obtained from Example 7.5 by taking *opposite quivers*, *opposite potentials*, and *opposite surface orientations*, and using the easy-to-show fact that QP-mutations and flips respectively commute with *taking opposites*.

8. FLIP \leftrightarrow QP-MUTATION COMPATIBILITY

Let (Σ, \mathbb{M}) be any surface different from a 5-punctured sphere. Thus, (Σ, \mathbb{M}) is allowed to be any of the surfaces listed in (6.2), (6.3) and (6.4).

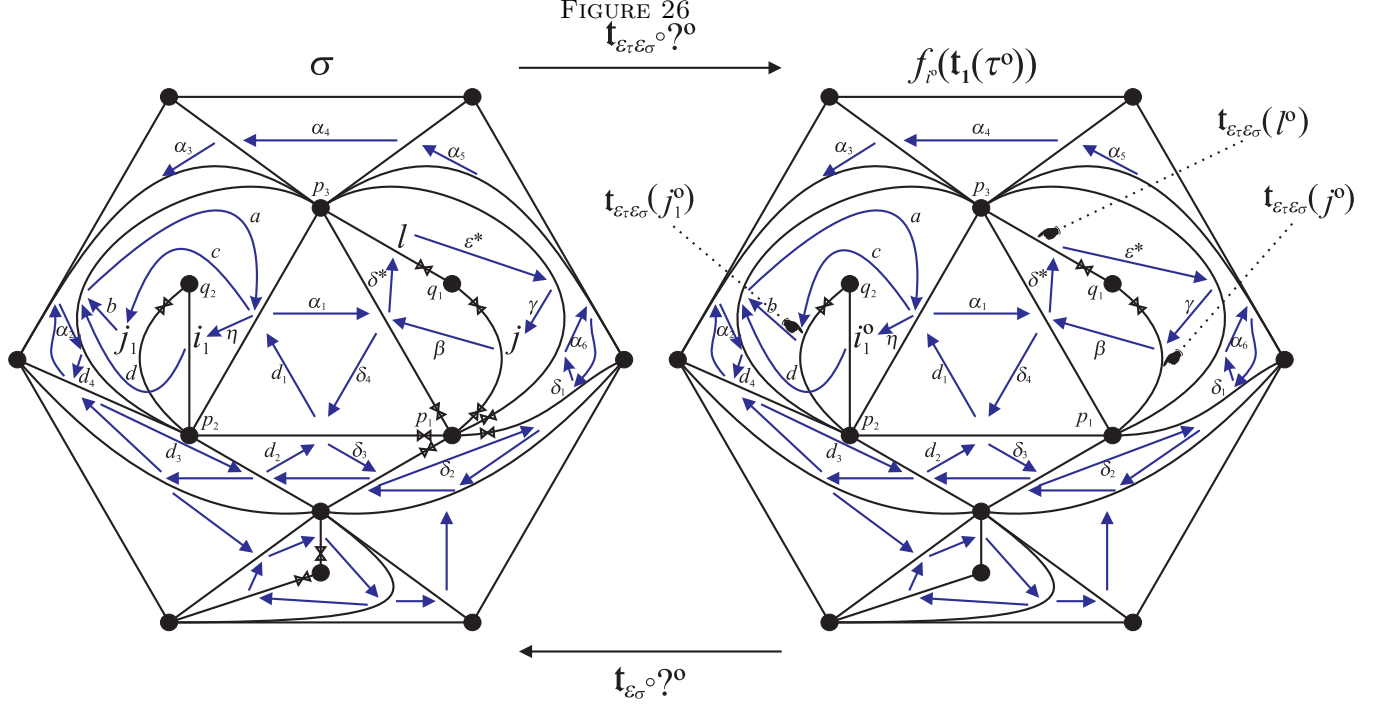
Theorem 8.1. *If τ and σ are tagged triangulations of (Σ, \mathbb{M}) related by the flip of a tagged arc $i \in \tau$, then the QPs $\mu_i(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\sigma), S(\sigma, \mathbf{x}))$ are right-equivalent.*

Proof. We have two possibilities: the weak signatures ϵ_τ and ϵ_σ either are equal, or they differ at exactly one puncture. Let $\mathbf{y} = (\epsilon_\tau(p)x_p)_{p \in \mathbb{P}}$.

Suppose $\epsilon_\tau = \epsilon_\sigma$, then $\sigma^\circ = f_{i^\circ}(\tau^\circ)$ (cf. Proposition 2.12). By Theorem 3.6, $\mu_{i^\circ}(Q(\tau^\circ), S(\tau^\circ, \mathbf{y}))$ is right-equivalent to $(Q(\sigma^\circ), S(\sigma^\circ, \mathbf{y}))$. Since $\mu_i(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\sigma), S(\sigma, \mathbf{x}))$ are respectively obtained from $\mu_{i^\circ}(Q(\tau^\circ), S(\tau^\circ, \mathbf{y}))$ and $(Q(\sigma^\circ), S(\sigma^\circ, \mathbf{y}))$ by the same relabeling of the vertices of the quiver $Q(\sigma^\circ)$, it follows that they are right-equivalent QPs.

Now suppose that ϵ_τ and ϵ_σ differ at exactly one puncture q . Without loss of generality we can assume that $\epsilon_\tau(q) = 1 = -\epsilon_\sigma(q)$. Then i° is a folded side of τ° incident to the puncture q , and $\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma^\circ) = f_{i^\circ}(\mathbf{t}_1(\tau^\circ))$. By Theorem 7.1, $\mu_{i^\circ}(Q(\mathbf{t}_1(\tau^\circ)), S(\mathbf{t}_1(\tau^\circ), \mathbf{y}))$ is right-equivalent to $(Q(\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma^\circ)), S(\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma^\circ), \mathbf{y}))$. Since $\mu_i(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\sigma), S(\sigma, \mathbf{x}))$ can be respectively obtained from $\mu_{i^\circ}(Q(\mathbf{t}_1(\tau^\circ)), S(\mathbf{t}_1(\tau^\circ), \mathbf{y}))$ and $(Q(\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma^\circ)), S(\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma^\circ), \mathbf{y}))$ by the same relabeling of the vertices of the quiver $Q(\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma^\circ))$, it follows that they are right-equivalent QPs. \square

If we flip the tagged arcs $i \in \tau$ and $i^\circ \in \mathbf{t}_1(\tau^\circ)$, we obtain the tagged triangulations $\sigma = f_i(\tau)$ and $\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma_\circ) = f_{i^\circ}(\mathbf{t}_1(\tau^\circ))$ depicted in Figure 26. Just as it happened with τ , the functions $\mathbf{t}_{\epsilon_\tau \epsilon_\sigma} \circ ?^\circ : \sigma \rightarrow$



$\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma_\circ) = f_{i^\circ}(\mathbf{t}_1(\tau^\circ))$ and $\mathbf{t}_\sigma \circ ?^\circ : \mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma_\circ) = f_{i^\circ}(\mathbf{t}_1(\tau^\circ)) \rightarrow \sigma$ act as mutually inverse relabelings, and induce mutually inverse quiver isomorphisms between $Q(\sigma)$ and $Q(\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma_\circ))$, which we denote by $\mathbf{t}_{\epsilon_\tau \epsilon_\sigma} \circ ?^\circ$ and $\mathbf{t}_\sigma \circ ?^\circ$ as well. Furthermore, just as we did with τ , we use $\mathbf{t}_{\epsilon_\tau \epsilon_\sigma} \circ ?^\circ$ and $\mathbf{t}_\sigma \circ ?^\circ$ to identify the arrow sets of $Q(\sigma)$ and $Q(\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma_\circ))$ (but not their vertex sets). Under this identification, we have

$$S(\sigma, \mathbf{x}) = S(\mathbf{t}_{\epsilon_\tau \epsilon_\sigma}(\sigma_\circ), \mathbf{y}) = abc + x_{q_1}^{-1} \epsilon^* \delta^* \beta \gamma - x_{q_2}^{-1} a d \eta - x_{p_1} \beta \gamma \delta_1 \delta_2 \delta_3 \delta_4 + x_{p_2} d_1 d_2 d_3 d_4 d \eta + x_{p_3} \epsilon^* \delta^* \alpha_1 a \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 + S'(\tau)$$

If we on the other hand apply the i^{th} QP-premutation to $(Q(\tau), S(\tau, \mathbf{x}))$, we obtain $(\tilde{\mu}_i(Q(\tau)), \tilde{\mu}_i(S(\tau, \mathbf{x})))$, where

$$\tilde{\mu}_i(S(\tau, \mathbf{x})) = \alpha \beta \gamma + abc - x_{q_1}^{-1} \alpha [\delta \epsilon] - x_{q_2}^{-1} a d \eta - x_{p_1} \delta_1 \delta_2 \delta_3 \delta_4 [\delta \epsilon] + x_{p_2} d_1 d_2 d_3 d_4 d \eta + x_{p_3} \alpha \alpha_1 a \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 + S'(\tau) + \epsilon^* \delta^* [\delta \epsilon]$$

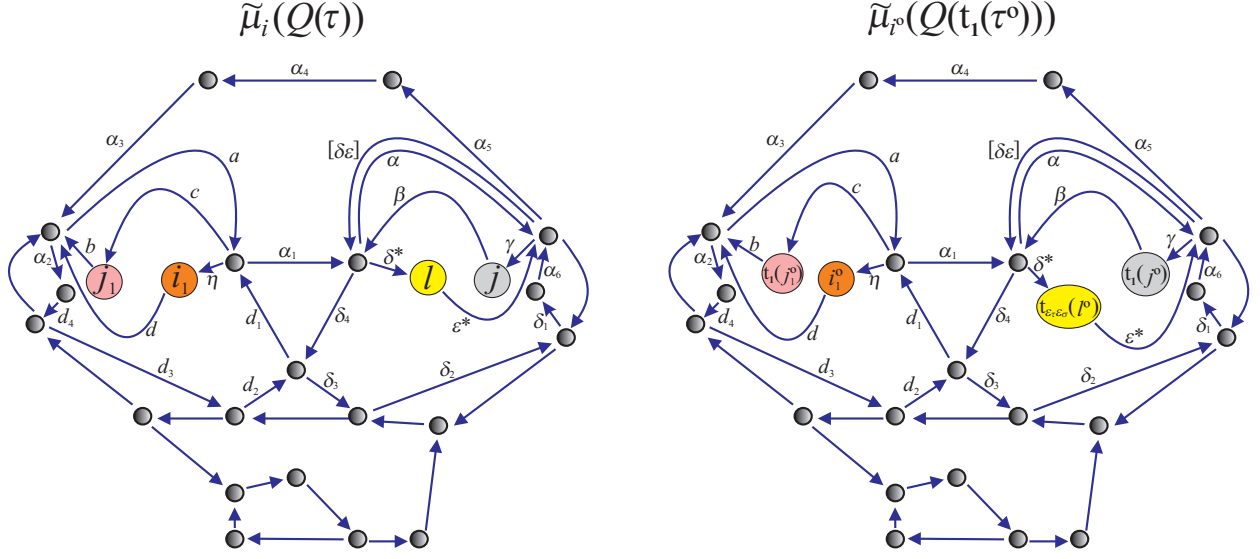
is a potential on the quiver $\tilde{\mu}_i(Q(\tau))$ depicted on the left side of Figure 27. Notice that the vertex $i \in \tau$ of $Q(\tau)$ has been replaced by the vertex $l \in \sigma$ of $\tilde{\mu}_i(Q(\tau))$. This is completely in sync with Fomin-Zelevinsky's *seed mutation* inside the *cluster algebra* associated to (Σ, \mathbb{M}) (with any *system of coefficients*, thanks to results of Fomin-Shapiro-Thurston); indeed, if we mutate the seed corresponding to τ in direction i , then (the cluster variable corresponding to) l replaces (the cluster variable corresponding to) i , not only at the level of clusters but also at the level of row and column labels of the associated skew-symmetric matrices $B(\tau)$ and $\mu_i(B(\tau)) = B(\sigma)$ (cf. [12, Definition 1.1] and the Definition of *seed* preceding it). The quiver counterpart of this replacement of row and column labels is precisely the replacement of i by l as a vertex of $\tilde{\mu}_i(Q(\tau))$ and $\mu_i(Q(\tau))$.

If we apply the $i^{\circ \text{th}}$ QP-premutation to $(Q(\mathbf{t}_1(\tau^\circ)), S(\mathbf{t}_1(\tau^\circ)), \mathbf{y})$ we obtain $(\tilde{\mu}_{i^\circ}(Q(\mathbf{t}_1(\tau^\circ))), \tilde{\mu}_{i^\circ}(S(\mathbf{t}_1(\tau^\circ)), \mathbf{y})))$, where

$$\tilde{\mu}_{i^\circ}(S(\mathbf{t}_1(\tau^\circ)), \mathbf{y}) = \alpha \beta \gamma + abc - x_{q_1}^{-1} \alpha [\delta \epsilon] - x_{q_2}^{-1} a d \eta - x_{p_1} \delta_1 \delta_2 \delta_3 \delta_4 [\delta \epsilon] + x_{p_2} d_1 d_2 d_3 d_4 d \eta + x_{p_3} \alpha \alpha_1 a \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 + S'(\tau) + \epsilon^* \delta^* [\delta \epsilon]$$

is a potential on the quiver $\tilde{\mu}_{i^\circ}(Q(\mathbf{t}_1(\tau^\circ)))$ depicted on the right side of Figure 27. Here, the vertex $i^\circ \in \mathbf{t}_1(\tau^\circ)$ of $Q(\mathbf{t}_1(\tau^\circ))$ has been replaced by the vertex $\mathbf{t}_{\epsilon_\sigma \epsilon_\tau}(l^\circ) \in \mathbf{t}_{\epsilon_\sigma \epsilon_\tau}(\tau^\circ)$ of $\tilde{\mu}_{i^\circ}(Q(\mathbf{t}_1(\tau^\circ)))$. This is again in sync with Fomin-Zelevinsky's *seed mutation* inside the *cluster algebra* associated to (Σ, \mathbb{M}) (with any system of coefficients).

FIGURE 27



Since Derksen-Weyman-Zelevinsky's QP-reduction does not touch quiver vertices at all, from the previous considerations we see that (8.1) is equivalent to the commutativity of the diagram (8.2). \blacktriangle

9. NON-DEGENERACY OF THE QPS $(Q(\tau), S(\tau, \mathbf{x}))$

The following is an immediate consequence of Theorem 8.1.

Corollary 9.1. *The QPs associated to tagged triangulations of (Σ, \mathbb{M}) are non-degenerate provided (Σ, \mathbb{M}) is not a sphere with less than 6 punctures, that is, provided (Σ, \mathbb{M}) is one of the following:*

- a surface with non-empty boundary, with or without punctures, and arbitrary genus;
- a positive-genus surface without boundary, and any number of punctures;
- a sphere with at least 6 punctures.

Thus, Conjecture 33 of [15] holds for all these surfaces.

Remark 9.2. (1) In [15, Theorem 31] it was shown that the QPs associated to ideal triangulations of (possibly punctured) surfaces with non-empty boundary are rigid, and this automatically implied their non-degeneracy by [10, Corollary 8.2]. Here we have presented a different proof of the non-degeneracy of these QPs: no allusion to rigidity has been made, and the potentials corresponding to tagged triangulations, missing in [15], have been calculated.

- (2) Since positive-genus closed surfaces with exactly one puncture do not possess ideal triangulations with self-folded triangles whatsoever, Theorem 3.6 (which is Theorem 30 of [15]) already implies the non-degeneracy of the QPs associated to triangulations of these surfaces, so the Popping Theorem is completely superfluous in this case.
- (3) Before the present paper was posted, the only empty-boundary surfaces for which the non-degeneracy of the QPs $(Q(\tau), S(\tau, \mathbf{x}))$ had been shown, are the positive-genus closed surfaces with exactly one puncture. Corollary 9.1 is therefore a strong improvement regarding the question of non-degeneracy, solving it in practically all cases.

10. IRRELEVANCE OF SIGNS AND SCALARS ATTACHED TO THE PUNCTURES

Let (Σ, \mathbb{M}) be any surface. A standard construction associates a *dimer model* $D(\sigma)$ to each ideal triangulation σ of (Σ, \mathbb{M}) without self-folded triangles. For the reader's convenience we give a quick description of this construction.

Pick an arbitrary point q_Δ inside each ideal triangle Δ of σ , in such a way that q_Δ does not lie on an arc of σ . Define a bipartite graph $D(\sigma)$ as follows:

- The vertex set of $D(\sigma)$ is $\mathbb{M} \cup \{q_\Delta \mid \Delta \text{ is an ideal triangle of } \sigma\}$;

- given $p \in \mathbb{M}$ and $q_\Delta \in \Delta$ with Δ ideal triangle of σ , connect p and q_Δ with an edge if and only if $p \in \Delta$;
- besides the edges just introduced, do not include more edges between vertices of $D(\sigma)$.

Since σ does not have self-folded triangles, we can suppose that the edges of $D(\sigma)$ have been drawn on the surface, and that whenever $p \in \mathbb{M}$ and $q_\Delta \in \Delta$ are connected by an edge of $D(\sigma)$, such edge is entirely contained in Δ and the only point in common it has with any arc of τ is p .

The straightforward proof of the following lemma is left to the reader.

Lemma 10.1. *Suppose $\partial\Sigma \neq \emptyset$. Then (Σ, \mathbb{M}) has an ideal triangulation τ without self-folded triangles such that every puncture of (Σ, \mathbb{M}) can be connected to a marked point lying on $\partial\Sigma$ through a simple walk on $D(\tau)$ (that is, a walk that does not repeat vertices of $D(\tau)$).*

The absence of self-folded triangles in τ allows us to draw the unreduced signed-adjacency quiver $\widehat{Q}(\tau)$ on (Σ, \mathbb{M}) as well. We notice then that $\widehat{Q}(\tau)$ is a subquiver of the *dimer dual* of $D(\tau)$.

We are now ready to show that the signs and scalars appearing in $S(\tau, \mathbf{x})$ are irrelevant whenever τ is a tagged triangulation of a surface with non-empty boundary. More precisely:

Proposition 10.2. *Let (Σ, \mathbb{M}) be a surface with non-empty boundary and let $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ and $\mathbf{y} = (y_p)_{p \in \mathbb{P}}$ be any two choices of non-zero scalars. Then for every tagged triangulation τ of (Σ, \mathbb{M}) , the QPs $(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\tau), S(\tau, \mathbf{y}))$ are right-equivalent.*

Proof. By Theorem 8.1, Lemma 10.1 and the fourth assertion of Proposition 2.7, we can assume, without loss of generality, that τ is as in the conclusion of Lemma 10.1.

Let $p_1, \dots, p_{|\mathbb{P}|}$, be any ordering of the punctures of (Σ, \mathbb{M}) . For $\ell = 0, 1, \dots, |\mathbb{P}|$, let $\mathbf{y}_\ell = (y_{\ell, p})_{p \in \mathbb{P}}$ be the choice of scalars defined as follows:

$$y_{\ell, p} = \begin{cases} 1 & \text{if } p \in \{p_1, \dots, p_\ell\}, \\ x_p & \text{otherwise.} \end{cases}$$

Since compositions of right-equivalences is right-equivalence, to prove the proposition it is sufficient to show that $(Q(\tau), S(\tau, \mathbf{y}_\ell))$ is right-equivalent to $(Q(\tau), S(\tau, \mathbf{y}_{\ell+1}))$ for $\ell = 0, \dots, |\mathbb{P}| - 1$ (notice that $(Q(\tau), S(\tau, \mathbf{y}_0)) = (Q(\tau), S(\tau, \mathbf{x}))$), and in order to do so, it is sufficient to show that $(\widehat{Q}(\tau), \widehat{S}(\tau, \mathbf{y}_\ell))$ is right-equivalent to $(\widehat{Q}(\tau), \widehat{S}(\tau, \mathbf{y}_{\ell+1}))$ (this is because reduced parts of QPs are uniquely determined up to right-equivalence).

Let $w = (q_1, e_1, q_2, e_2, \dots, q_{t-1}, e_{t-1}, q_t)$ be a simple walk on $D(\tau)$ such that

- $q_1 = p_{n+1}$,
- q_t lies on $\partial\Sigma$;
- none of the vertices q_1, \dots, q_{t-1} of $D(\tau)$ lies on $\partial\Sigma$.

Each edge e_s , $s = 1, \dots, t-2$, uniquely determines an arrow α_s of $\widehat{Q}(\tau)$, whereas e_{t-1} may or may not do so. In any case, all arrows determined by these edges are pairwise distinct. This implies that the R -algebra automorphism φ of $R\langle\langle\widehat{Q}(\tau)\rangle\rangle$ defined by

$$\varphi(\alpha_s) = \begin{cases} x_{p_{n+1}}^{-1} \alpha_s & \text{if } s \text{ is odd,} \\ x_{p_{n+1}} \alpha_s & \text{if } s \text{ is even,} \end{cases}$$

is a right-equivalence $\varphi : (\widehat{Q}(\tau), \widehat{S}(\tau, \mathbf{y}_\ell)) \rightarrow (\widehat{Q}(\tau), \widehat{S}(\tau, \mathbf{y}_{\ell+1}))$. □

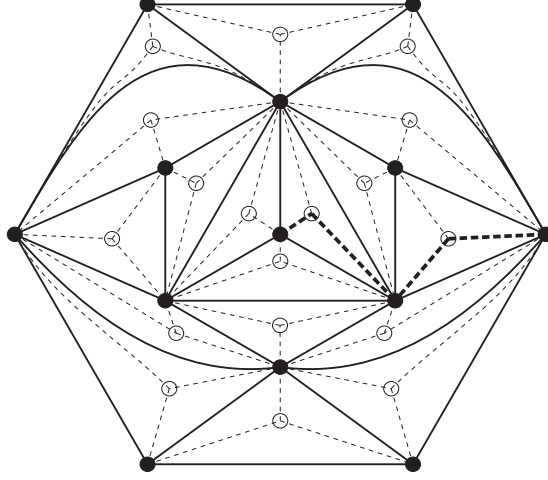
Example 10.3. In Figure 28 we can see an ideal triangulation τ , with the bipartite graph $D(\tau)$ drawn on the surface as well. The walk on $D(\tau)$ that consists of the edges of $D(\tau)$ that have been drawn bolder, helps to get rid of the scalar attached to the puncture in the center of the surface. ▲

For surfaces with empty boundary, we have the following :

Proposition 10.4 ([14]). *Let (Σ, \mathbb{M}) be a surface with empty boundary and let $\mathbf{x} = (x_p)_{p \in \mathbb{P}}$ and $\mathbf{y} = (y_p)_{p \in \mathbb{P}}$ be any two choices of non-zero scalars. Then for every tagged triangulation τ of (Σ, \mathbb{M}) , the QPs $(Q(\tau), S(\tau, \mathbf{x}))$ and $(Q(\tau), \lambda S(\tau, \mathbf{y}))$ are right-equivalent for some non-zero scalar λ .*

Remark 10.5. The original motivation for working with arbitrary choices $(x_p)_{p \in \mathbb{P}}$ of non-zero scalars, rather than with the particular choices $\mathbf{1} = (1)_{p \in \mathbb{P}}$ and $-\mathbf{1} = (-1)_{p \in \mathbb{P}}$, came from the wish of being able to obtain as many non-degenerate potentials as possible.

FIGURE 28. Getting rid of scalars attached to the punctures



11. JACOBIAN ALGEBRAS ARE FINITE-DIMENSIONAL

It is natural to ask whether the Jacobian algebras of the QPs arising from (tagged) triangulations are finite-dimensional or not. For polygons with at most one puncture, the answer was known to Caldero, Chapoton and Schiffler (cf. [6] and [7]) and Schiffler (cf. [21]), although these authors did not use the language of potentials and Jacobian algebras in the referred papers. For more general surfaces, the first partial answers to the Jacobi-finiteness question were given in [3], [15] and [16].

Assem, Brüstle, Charbonneau-Jodoin and Plamondon proved:

Theorem 11.1 ([3]). *If (Σ, \mathbb{M}) is an unpunctured surface, then all QPs of ideal triangulations of (Σ, \mathbb{M}) have finite-dimensional Jacobian algebras.*

The following more general result was simultaneously found by the author.

Theorem 11.2 ([15, Theorem 36]). *If the surface (Σ, \mathbb{M}) has non-empty boundary, then for every ideal triangulation τ of (Σ, \mathbb{M}) the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau, \mathbf{x}))$ is finite-dimensional.*

Example 11.3 ([16, Example 8.2]). Let τ be any ideal triangulation of the once-punctured torus (with empty boundary), then the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau, \mathbf{x}))$ is finite-dimensional.

Trepode and Valdivieso-Díaz have recently shown:

Theorem 11.4 ([22]). *Let (Σ, \mathbb{M}) be a sphere with $n \geq 5$ punctures. Then for any ideal triangulation τ of (Σ, \mathbb{M}) , the Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau, \mathbf{x}))$ is finite-dimensional.*

Ladkani proved the following generalization.

Theorem 11.5 ([17]). *The Jacobian algebra $\mathcal{P}(Q(\tau), S(\tau, \mathbf{x}))$ is finite-dimensional for every ideal triangulation τ of a surface with empty boundary.*

Since finite-dimensionality of Jacobian algebras is preserved by QP-mutations (cf. [10, Corollary 6.6]), a straightforward combination of Theorem 8.1 with Theorems 11.1, 11.2, 11.4 and 11.5, yields:

Corollary 11.6. *All the QPs associated to tagged triangulations of the surfaces considered in the present note have finite-dimensional Jacobian algebras.*

Remark 11.7. This last corollary does not imply that a Jacobian algebra of the form $\mathcal{P}(Q(\tau), S(\tau, \mathbf{x}))$ is necessarily isomorphic to the quotient $R\langle Q(\tau) \rangle / J_0(S(\tau, \mathbf{x}))$ (by definition, $J_0(S(\tau, \mathbf{x}))$ is the ideal of $R\langle Q(\tau) \rangle$ generated by the cyclic derivatives of $S(\tau, \mathbf{x})$), nor that $J_0(S(\tau, \mathbf{x}))$ is an admissible ideal of $R\langle Q(\tau) \rangle$ (cf. [9, Remark 5.2]). Indeed, if the underlying surface has empty boundary, then $R\langle Q(\tau) \rangle / J_0(S(\tau, \mathbf{x}))$ is infinite-dimensional, while $\mathcal{P}(Q(\tau), S(\tau, \mathbf{x}))$ is not. On the other hand, in the non-empty boundary situation, $J_0(S(\tau, \mathbf{x}))$ is always an admissible ideal of $R\langle Q(\tau) \rangle$, and $\mathcal{P}(Q(\tau), S(\tau, \mathbf{x}))$ is always isomorphic to $R\langle Q(\tau) \rangle / J_0(S(\tau, \mathbf{x}))$ (cf. [3] and [9, Theorem 5.5]).

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